INTERSECTION BOUNDS FOR NODAL SETS OF PLANAR NEUMANN EIGENFUNCTIONS WITH INTERIOR ANALYTIC CURVES

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ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded piecewise smooth domain and φ_λ be a Neumann (or Dirichlet) eigenfunction with eigenvalue λ^2 and nodal set $\mathcal{N}_{\varphi_\lambda} = \{x \in \Omega; \varphi_\lambda(x) = 0\}$. Let $H \subset \Omega$ be an interior C^ω curve. Consider the intersection number

$$n(\lambda, H) := \#(H \cap \mathcal{N}_{\varphi_\lambda}).$$

We first prove that for general piecewise-analytic domains, and under an appropriate "goodness" condition on H (see Theorem 1.1),

$$n(\lambda, H) = \mathcal{O}_H(\lambda) \ (*)$$

as $\lambda \to \infty$. Then, using Theorem 1.1, we prove in Theorem 1.2 that the bound in $(*)$ is satisfied in the case of quantum ergodic (QE) sequences of interior eigenfunctions, provided Ω is convex and H has strictly positive geodesic curvature.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a real analytic, bounded planar domain with boundary $\partial\Omega$ and $H \subset \mathring{\Omega}$ a real-analytic interior curve. We consider here the Neumann (or Dirichlet) eigenfunctions φ_λ on real analytic plane domains $\Omega \subset \mathbb{R}^2$ with

$$\begin{cases} -\Delta\varphi_\lambda = \lambda^2\varphi_\lambda & \text{in } \Omega \\ \partial_\nu\varphi_\lambda = 0 \ (Neumann), \quad \varphi_\lambda = 0 \ (Dirichlet) & \text{on } \partial\Omega. \end{cases}$$

The nodal set of φ_λ is by definition

$$N_{\varphi_\lambda} = \{x \in \Omega : \varphi_\lambda(x) = 0\}.$$

Our main interest here involves estimating from above the number of intersection points of the nodal lines of Neumann eigenfunctions (the connected components of the nodal set) with a fixed analytic curve H contained in the interior of the domain Ω. We define the intersection number for Dirichlet data along H by

$$n(\lambda, H) = \#\{N_{\varphi_\lambda} \cap H\}. \tag{1}$$

We recall from [TZ] that an interior curve H is said to be *good* provided for some $\lambda_0 > 0$ there is a constant $C = C(\lambda_0) > 0$ such that for all $\lambda \geq \lambda_0$,

$$\int_H |\varphi_\lambda|^2 d\sigma \geq e^{-C\lambda}. \tag{2}$$

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Assuming the goodness condition (2), it is proved in [TZ] that

$$n(\lambda, H) = \mathcal{O}_H(\lambda). \tag{3}$$

It follows from unique continuation for the interior eigenfunctions and the potential layer formula $\varphi_\lambda(x) = \int_{\partial\Omega} N(x, r(s); \lambda)\varphi_\lambda(s) d\sigma(s);\ x \in \text{int}(\Omega)$, that (2) is satisfied in the special case where $H = \partial\Omega$. The goodness property (2) seems very likely generic (see [BR]). However, it is difficult to prove in concrete examples that the same upper bound is satisfied for all eigenfunctions with $\lambda \geq \lambda_0$. Indeed, in [TZ], only the special curve $\partial\Omega$ is shown to be good. Recently, Jung [Ju] has shown that in the boundaryless case, closed horocycles of hyperbolic surfaces of finite volume are good in the sense of (2) and hence satisfy the $\mathcal{O}(\lambda)$ upper bounds. In the case of flat 2-torus, Bourgain and Rudnick [BR] have recently proved $\lessapprox \lambda$ upper bounds when H is real-analytic with nowhere vanishing curvature (they also prove $\gtrapprox \lambda^{1-\varepsilon}$ lower bounds in the case where H is real-analytic and non-geodesic).

Despite these results, it is clear that not all curves are good in the sense of (2). As a counterexample, consider the Neumann problem in the unit disc. The eigenfunctions in polar variables $(r, \theta) \in (0, 1] \times [0, 2\pi]$ are $\varphi^{even}_{m,n}(r, \theta) = C_{m,n} \cos m\theta J_m(j'_{m,n} r)$ and $\varphi^{odd}_{m,n}(r, \theta) = C_{m,n} \sin m\theta J_m(j'_{m,n} r)$. Here, J_m is the m-th integral Bessel function and $j'_{m,n}$ is the m-th critical point of J_m. The eigenvalues are $\lambda^2_{m,n} = (j'_{m,n})^2$. Fix $m \in \mathbb{Z}^+$ and consider

$$H_m = \{(r, \theta); \theta = \frac{2\pi k}{m};\ k = 0, ..., m - 1\}.$$

Then, clearly for any $n = 0, 1, 2, ...$ $\varphi^{odd}_{m,n}|_{H_m} = 0$ and so in particular H_m is not good in the sense of (2).

The point of this paper is twofold:

(i) to weaken the goodness assumption needed for the $\mathcal{O}_H(\lambda)$ intersection bound (see Theorem 1.1),
(ii) to use the new goodness condition to explicitly identify a large class of *interior* analytic curves in ergodic billiards that satisfy the $n(\lambda, H) = \mathcal{O}_H(\lambda)$ intersection bounds for interior quantum ergodic (QE) sequences of eigenfunctions. That is the content of Theorem 1.2.

Moreover, for both Theorems 1.1 and 1.2, the $n(\lambda, H)$ upper bound is proved using the frequency function method of F.H. Lin combined with some semiclassical microlocal analysis, rather than the Jensen argument in [TZ]. Indeed, the revised goodness condition (see Theorem 1.1) that is needed for all our results follows here readily from the main frequency function bound for the number of complex zeros in a complex thickening of H.

Our first main theorem is:

Theorem **1.1.** *Let Ω be a bounded, piecewise-analytic domain and $H \subset \mathring{\Omega}$ an interior, C^ω curve with restriction map $\gamma_H : C^0(\Omega) \to C^0(H)$. Let $H^{\mathbb{C}}_{\varepsilon_\circ}$ denote the complex radius $\varepsilon_\circ > 0$ Grauert tube containing H as its totally real submanifold and $(\gamma_H \varphi_\lambda)^{\mathbb{C}}$ be the holomorphic continuation of $\gamma_H \varphi_\lambda$ to $H^{\mathbb{C}}_{\varepsilon_\circ}$. Suppose the curve H satisfies the* revised goodness condition

$$\sup_{z \in H^{\mathbb{C}}_{\varepsilon_\circ}} |(\gamma_H \varphi_\lambda)^{\mathbb{C}}(z)| \geq e^{-C\lambda}\ (*)$$

for some $C > 0$. Then, there is a constant $C_{\Omega,H} > 0$ such that for all $\lambda \geq \lambda_0$,

$$n(\lambda, H) \leq C_{\Omega,H}\lambda.$$

Our results here are inherently semiclassical and so we introduce the parameter h which takes values in the sequence $\lambda_j^{-1}; j = 1, 2, 3, ...$ By a slight abuse of notation, we denote the Neumann (or Dirichlet) eigenfunctions φ_λ by φ_h, and write $n(h, H) := n(\lambda, H)$. The restrictions to H are denoted by $\varphi_h^H := \gamma_H \varphi_h$ where $\gamma_H : C^0(\Omega) \to C^0(H)$ is the restriction operator $\gamma_H f = f|_H$. In the special case where $H = \partial\Omega$ we denote the Neumann (resp. Dirichlet) boundary traces by $\varphi_h^{\partial\Omega} := \gamma_{\partial\Omega}\varphi_h$ (resp. $\varphi_h^{\partial\Omega} := \gamma_{\partial\Omega} h\partial_\nu \varphi_h$).

Our second results deals with the case of quantum ergodic sequences of eigenfunctions. We recall that given a piecewise smooth manifold Ω with boundary, a sequence of L^2-normalized eigenfunctions $(\varphi_{h_{j_k}})_{k=1}^\infty$ is *quantum ergodic* (QE) if for any $a \in S^0(T^*\Omega)$ with $\pi(\mathrm{supp}(a)) \subset \mathrm{Int}(\Omega)$,

$$\langle Op_{h_{j_k}}(a)\varphi_{h_{j_k}}, \varphi_{h_{j_k}}\rangle \sim_{h_{j_k}\to 0^+} \int_{S^*\Omega} a(x,\xi)d\mu,$$

where $d\mu$ is Liouville measure. By a theorem of Zelditch and Zworski [ZZ], for a domain with ergodic billiards, a density-one subset of eigenfunctions are quantum ergodic. The domain Ω is *quantum uniquely ergodic* (QUE) if all subsequences are QE.

An important consequence of Theorem 1.1 concerns convex ergodic billiards.

THEOREM **1.2.** *Let Ω be a bounded, piecewise-analytic convex domain with ergodic billiard flow and H be a C^ω interior curve with strictly positive geodesic curvature. Let $(\varphi_{h_{j_k}})_{k=1}^\infty$ be a QE sequence of Neumann or Dirichlet eigenfunctions in Ω. Then,*

$$n(h_{j_k}, H) = \mathcal{O}_{H,\Omega}(h_{j_k}^{-1}).$$

Our nodal intersection bounds are consistent with S.T. Yau's famous conjecture on the Hausdorff measure of nodal sets [BG, Do, DF, DF2, H, HL, HHL, HS, L, Y1, Y2] which asserts that for all smooth (M, g) there are constants $c_1, C_1 > 0$ such that $c_1\lambda \leq |N_{\varphi_\lambda}| \leq C_1\lambda$, where $|\cdot|$ denotes Hausdorff measure. There has been important recent progress on polynomial lower bounds in Yau's conjecture using several methods (see [CM], [He], [Man], [SZ]). Contrary to the lower bounds on nodal length, there are no general nontrivial lower bounds for the intersection count studied here which is easily seen by considering the disc (see also [JN, NJT, NS] and related results on sparsity of nodal domains [Lew]). In analogy with the case of nodal domains, it is of interest to determine whether non-trivial (ie. polynomial in λ) lower bounds exist for nodal intersections under appropriate dynamical assumptions (such as ergodicity) on the billiard dynamics. Recently, in [GRS], Ghosh, Reznikov and Sarnak have established such polynomial lower bounds in the case of arithmetic surfaces. We hope to return to this question elsewhere.

Throughtout the paper $C > 0$ will denote a positive constant that can vary from line to line.

1.1. **Outline of the proof of Theorem 1.1.** We now describe the main ideas in the proof of Theorem 1.1 suppressing for the moment some of the technicalities. Let $q : [-\pi, \pi] \to H$ be a C^ω-parametrization of the curve H with $|q'(t)| \neq 0$ for all $t \in [-\pi, \pi]$ and let $r : [-\pi, \pi] to \partial\Omega$ be the arclength parametrization of the boundary. We denote the respective eigenfunction restrictions (on the parameter domain) by $u_h^H(t) = \varphi_h^H(q(t))$ and $u_h^{\partial\Omega}(s) = \varphi_h^{\partial\Omega}(r(s))$. As

in [TZ], given the eigenfunction restriction, $u_h^H(t) = \varphi_h^H(q(t))$, $t \in [-\pi,\pi]$ the first step is to complexify u_h^H to a holomorphic function $u_h^{H,\mathbb{C}}(t)$ with $t \in C_{\varepsilon_\circ}$ where $C_{\varepsilon_\circ}$ is a simply-connected domain with C^ω boundary $\partial C_{\varepsilon_\circ}$ containing the rectangle $S_{\varepsilon_\circ,\pi}$ in the parameter space. The reason for introducing the intermediate domain of holomorphy $C_{\varepsilon_\circ}$, is somewhat technical and has to do with the frequency function approach to nodal estimates, which is adapted to counting complex zeros in discs (see Lemma 3.3). Let $n(h, C_{\varepsilon_\circ})$ denotes the number of complex zeros of $u_h^{H,\mathbb{C}}$ in the simply connected domain $C_{\varepsilon_\circ}$. The key frequency function estimate (see Proposition 3.4) gives the upper bound

$$n(h,H) \leq n(h, C_{\varepsilon_\circ}) \leq C_1 \left(\frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} \right). \tag{4}$$

Here, we write $L^2_{\varepsilon_\circ}$ for $L^2(\partial C_{\varepsilon_\circ}, d\sigma(t))$ and ∂_T is the unit tangential derivative along $\partial C_{\varepsilon_\circ}$. A key step in the proof of Theorem 1.1 is to h-microlocally decompose the right hand side in (4). Let $\chi_R \in C_0^\infty(T^*\partial C_{\varepsilon_\circ})$ with $\chi_R(\sigma) = 1$ for $|\sigma| \leq R+1$ and $\chi_R(\sigma) = 0$ for $|\sigma| \geq R+2$ with $R > 0$ arbitrary. Clearly,

$$\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} \leq \|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} + \|\partial_T (1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}. \tag{5}$$

For the first term on the right hand side of (5), since $h\partial_T Op_h(\chi_R) \in Op_h(S^{0,0}(T^*\partial C_{\varepsilon_\circ}))$, we have by L^2-boundedness that

$$\frac{\|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = h^{-1} \frac{\|h\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} \leq C_2 h^{-1}. \tag{6}$$

As for the second term on the right hand side of (5), by using potential layer formulas and the Cauchy-Schwarz inequality combined with a complex contour deformation argument (see Proposition 4.2), we show that

$$\|h\partial_T (1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} = \mathcal{O}(e^{-C_R/h}) \cdot \|u_h^{\partial\Omega}\|_{L^2}. \tag{7}$$

Here, $C_R \gtrapprox R$ as $R \to \infty$ and $L^2_0 = L^2([-\pi,\pi], dt)$, so the term on the right hand side of (7) involves the L^2-integral of the restriction of φ_h to the domain boundary $\partial\Omega$.
Since $\|u_h^{\partial\Omega}\|_{L^2} = O(h^{-\alpha})$ for some $\alpha > 0$, it follows that

$$\|h\partial_T (1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} = \mathcal{O}(e^{-C_R/h}). \tag{8}$$

From the Cauchy integral formula, Cauchy-Schwarz and the goodness condition $(*)$ we get

$$\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} \geq C \cdot \sup_{t \in S_{\varepsilon_\circ,\pi}} |u_h^{H,\mathbb{C}}(t)| \gtrapprox e^{-C_0/h}. \tag{9}$$

From (9) and (8),

$$\frac{\|h\partial_T (1 - Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = \mathcal{O}(e^{(-C_R + C_0)/h}). \tag{10}$$

By choosing R sufficiently large in the radial frequency cutoff χ_R, we get that $C_R - C_0 \gtrapprox R > 0$ and so, substitution of the estimates (5), (6) and (10) in (4) completes the proof of Theorem 1.1. $\square$

1.2. **Outline of the proof of Theorem 1.2.** Let $H^{\mathbb{C}}(\varepsilon_\circ)$ be a complex Grauert tube of radius $\varepsilon_\circ > 0$ with totally-real part H and $\zeta_{\varepsilon_\circ} \in C^\infty(H^{\mathbb{C}}(\varepsilon_\circ); [0,1])$ be a cutoff on the Grauert tube equal to 1 on the annulus $H^{\mathbb{C}}(2\varepsilon_\circ/3) - H^{\mathbb{C}}(\varepsilon_\circ/3)$ and vanishing outside $H^{\mathbb{C}}(5\varepsilon_\circ/6) - H^{\mathbb{C}}(\varepsilon_\circ/6)$. Let $\chi_{\varepsilon_\circ}(t) := \zeta_{\varepsilon_\circ}(q^{\mathbb{C}}(t))$ be the corresponding cutoff in the parameter domain.

Ignoring technicalities arising from corner points on the boundary, the main technical part of the proof of Theorem 1.2 (see Proposition 8.1) consists of showing that under the non-vanishing curvature condition on H and for $\varepsilon_\circ > 0$ small, there is an order-zero semiclassical pseudodifferential operator $P(h) \in Op_h(C_0^\infty(B^*\partial\Omega))$ and a subharmonic weight function $S \in C^\omega(\operatorname{supp}\chi_{\varepsilon_\circ}; \mathbb{R}^+)$ such that

$$h^{-1/2} \int\int_{\mathbb{C}/2\pi\mathbb{Z}} e^{-2S(t)/h} |u_h^{H,\mathbb{C}}(t)|^2 \, \chi_{\varepsilon_\circ}(t)\, dt d\bar{t} \sim_{h\to 0^+} \langle P(h)\varphi_h^{\partial\Omega}, \varphi_h^{\partial\Omega}\rangle. \tag{11}$$

Moreover, the principal symbol $\sigma(P(h))$ satisfies

$$\int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dy d\eta \geq C_{H,\Omega,\varepsilon_\circ} > 0 \tag{12}$$

where $\gamma(y,\eta) = \sqrt{1-|\eta|^2}$.

Given a quantum ergodic sequence $(\varphi_{h_{j_k}})_{k=1}^\infty$, it follows that the boundary restrictions $(\varphi_{h_{j_k}}^{\partial\Omega})_{k=1}^\infty$ are themselves quantum ergodic in the sense that

$$\langle P(h)\varphi_h^{\partial\Omega}, \varphi_h^{\partial\Omega}\rangle \sim_{h\to 0^+} \int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dy d\eta. \tag{13}$$

It then follows from (11), (12) and (13) that

$$h^{-1/2} \int\int_{\mathbb{C}/2\pi\mathbb{Z}} e^{-2S(t)/h} |u_h^{H,\mathbb{C}}(t)|^2 \, \chi_{\varepsilon_\circ}(t)\, dt d\bar{t} \sim_{h\to 0^+} \int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dy d\eta = C_{\Omega,H,\varepsilon_\circ} > 0. \tag{14}$$

From the lower bound in (14), the revised goodness condition $(*)$ must be satisfied and Theorem 1.2 then follows from Theorem 1.1. □

Remark: In (13) we have used that for Dirichlet, interior QUE for domains implies QUE for the boundary traces $\varphi_h^{\partial\Omega}$. This follows from Burq's proof of boundary quantum ergodicity [Bu] using the Rellich commutator argument (see also [HZ] for a different proof) . Similarily, for Neumann, the same is true as long as one uses test operators with symbols supported away from the tangential set to the boundary; in particular, our test operator $P(h)$ in (11) has this property. Neither statement is necessarily correct for the eigenfunction restrictions to a general interior curve H [TZ2]. An important point in this paper is that the nodal intersection count for an interior H is linked to QER for the boundary values of eigenfunctions $\varphi_h^{\partial\Omega}$ (not the QER problem for H). Indeed, the identity (11) directly links a weighted L^2-integral of the holomorphic eigenfunction continuations over H to boundary QER. That part of the argument is quite technical and uses the curvature assumption on H in a crucial way (see section 8).

Remark: Recently, Zelditch [Z] has obtained detailed results on the asymptotic distribution of complex zeros of $\varphi_h^{H,\mathbb{C}}$ in the ergodic case when H is a geodesic. Although we do not pursue this here, the identity in (14) can be used to derive asymptotic distribution results for complex zeros of $\varphi_h^{H,\mathbb{C}}$ in the case where H has strictly positive geodesic curvature, but

only in an annular subdomain of $H_{\varepsilon_\circ}^{\mathbb{C}}$ away from the real curve H (ie. on the support of the cutoff $\chi_{\varepsilon_\circ}$). At the moment, we do not know what the asymptotic distribution of the zeros of $\varphi_h^{H,\mathbb{C}}$ looks like in the entire Grauert tube $H_{\varepsilon_\circ}^{\mathbb{C}}$ when H is geodesically curved. We hope to return to this problem elsewhere.

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2. Analytic continuation of eigenfunctions and domains

2.1. **Complexification of domains Ω and their boundaries $\partial\Omega$.** We adopt notation that is similar to that of Garabedian [G] and Millar [M1, M2] (see also [TZ]) and denote points in $\mathbb{R}^2$ by (x, y) and complex coordinates in $\mathbb{C}^2$ by (z, w). It is also important to single out the independent complex coordinates $\zeta = z + iw, \zeta^* = z - iw$. When $H \subset \Omega$ and $\partial\Omega$ are real analytic curves, their complexifications are the images of analytic continuations of real analytic parameterizations. There are two natural parameter spaces and, as in [TZ], we freely work with both throughout. We define the parameter strip of width $2\varepsilon_0$ to be

$$S_{\varepsilon_\circ} = \{t \in \mathbb{C} : t = \mathrm{Re}\, t + i\mathrm{Im}\, t, t \in \mathbb{R}, \mathrm{Im}\, t \in [-\varepsilon_\circ, \varepsilon_\circ]\}.$$

The corresponding fundamental rectangular domain is

$$S_{\varepsilon_\circ,\pi} = \{t \in \mathbb{C} : t = \mathrm{Re}\, t + i\mathrm{Im}\, t, \mathrm{Re}\, t \in [-\pi, \pi], \mathrm{Im}\, t \in [-\varepsilon_\circ, \varepsilon_\circ]\}.$$

For $\varepsilon_\circ > 0$ small, the associated conformal map of $S_{\varepsilon_\circ,\pi}$ onto $H_{\varepsilon_\circ}^{\mathbb{C}}$ is

$$q^{\mathbb{C}} : S_{\varepsilon_\circ,\pi} \longrightarrow H_{\varepsilon_\circ}^{\mathbb{C}}$$
$$q^{\mathbb{C}}(t) = (q_1^{\mathbb{C}}(t), q_2^{\mathbb{C}}(t)).$$

Without loss of generality, we assume that H is a closed curve with $|q'(t)| \neq 0$ for all $t \in [-\pi, \pi]$. In addition, we assume throughout that the real-analytic parametrization $q : [-\pi, \pi] \to H$ with $q(t + 2\pi) = q(t)$ extends to a conformal map $q^{\mathbb{C}} : S_{2\varepsilon_\circ,2\pi} \to H_{2\varepsilon_\circ}^{\mathbb{C}}$ with $q^{\mathbb{C}}(t + 2\pi) = q^{\mathbb{C}}(t)$. One can also naturally parameterize $H_{\varepsilon_\circ}^{\mathbb{C}}$ using functions on annular domains in $\mathbb{C}$ of the form

$$A_{\varepsilon_\circ} := \{z \in \mathbb{C}; e^{-\varepsilon_\circ} \leq |z| \leq e^{\varepsilon_\circ}\}.$$

In terms of the conformal map

$$z : S_{\varepsilon_\circ,\pi} \longrightarrow A_{\varepsilon_\circ},\ z(t) = e^{it},$$

given any holomorphic function $f \in \mathcal{O}(S_{\varepsilon_\circ,\pi})$ there is a unique holomorphic $F \in \mathcal{O}(A_{\varepsilon_\circ})$ with

$$f(t) = F(z(t)) = F(e^{it}).$$

The conformal parametrizing map $q^{\mathbb{C}} : S_{\varepsilon_\circ,\pi} \to H_{\varepsilon_\circ}^{\mathbb{C}}$ induces a conformal parametrizing map $Q^{\mathbb{C}} : A_{\varepsilon_\circ} \to H_{\varepsilon_\circ}^{\mathbb{C}}$ with $q^{\mathbb{C}}(t) = Q^{\mathbb{C}}(e^{it})$. We use the two maps interchangeably throughout. Generally, upper case letters denote parametrization maps from the annulus $A_{\varepsilon_\circ}$ and lower case ones denote maps form the rectangle $S_{\varepsilon_\circ,\pi}$. In view of the potential layer formulas and the boundary conditions, the boundary curve $\partial\Omega$ has special significance. Without loss of generality, we let $r : [-\pi, \pi] \to \partial\Omega$ be the real analytic arclength parametrization of the boundary with $r(t + 2\pi) = r(t)$ and $|r'(t)| = 1$ for all $t \in [-\pi, \pi]$. The corresponding holomorphic continuation is $r^{\mathbb{C}} : S_{\varepsilon_\circ,\pi} \longrightarrow \partial\Omega_{\varepsilon_\circ}^{\mathbb{C}}$ with $r^{\mathbb{C}}(t) = R^{\mathbb{C}}(z(t))$.

In addition, we let $C_{\varepsilon_\circ}$ be a simply-connected domain bounded by a closed real-analytic curve $\partial C_{\varepsilon_\circ}$ with

$$[-\pi,\pi] \subsetneqq S_{\varepsilon_\circ,\pi} \subsetneqq C_{\varepsilon_\circ} \subsetneqq S_{2\varepsilon_\circ,2\pi}, \tag{15}$$

and

$$\min_{z\in\partial C_{\varepsilon_\circ}\cap\mathbb{R}} |z-[-\pi,\pi]| \geq \frac{\pi}{2} \text{ and } \max_{z\in\partial C_{\varepsilon_\circ}} |\operatorname{Im} z| \leq \frac{7\varepsilon_\circ}{4}.$$

The interval $[-\pi,\pi]$ is just the totally real slice of the complex parameter rectangle $S_{\varepsilon_\circ,\pi}$ which is contained in $C_{\varepsilon_\circ}$. By possibly shrinking $\varepsilon_\circ > 0$ we assume from now on that the eigenfunction restrictions extend to 2π-real periodic holomorphic functions $u_h^{H,\mathbb{C}}$ on the larger rectangles $S_{2\varepsilon_\circ,2\pi}$.

2.1.1. *Holomorphic continuation of the restricted eigenfunctions.* Let $G : H^{-2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ with Schwartz kernel

$$G(x,y,x',y',h) = \frac{i}{4}\mathrm{Ha}_0^{(1)}(h^{-1}|(x,y)-(x',y')|),$$

where

$$\mathrm{Ha}_\nu^{(1)}(z) = c_\nu \frac{e^{iz}}{\sqrt{z}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}}(1-\frac{s}{2iz})^{\nu-\frac{1}{2}} ds, \quad \operatorname{Re} z > 0. \tag{16}$$

An application of Green's theorem yields the following potential layer formula for the Neumann eigenfunctions

$$\varphi_h(x,y) = \int_{\partial\Omega} \partial_{\nu_s} G(x,y;r(s),h)\, \varphi_h(r(s)) d\sigma(s), \tag{17}$$

where $(x,y) \in \mathring{\Omega}$ and $\nu_s \in S_{\partial\Omega}(\Omega)$ is the unit external normal to the boundary at $r(s) \in \partial\Omega$. We denote the kernel of the potential layer operator in (17) by

$$N(x,y;r(s),h) := \partial_{\nu_s} G(x,y;r(s),h) = -h^{-1}\mathrm{Ha}_1^{(1)}(h^{-1}|(x,y)-r(s)|)\cos\theta((x,y),r(s)) \tag{18}$$

where

$$\cos\theta((x,y),r(s)) = \left\langle \frac{(x,y)-r(s)}{|(x,y)-r(s)|}, \nu_s \right\rangle$$

and the corresponding operator by $N(h) : C^\infty(\partial\Omega) \to C^\infty(\mathring{\Omega})$.

To understand holomorphic continuation of eigenfunctions, one starts with the singularity decomposition of the kernel $G(x,y;r(s),h)$. It is well-known that

$$G(x,y;r(s),h) = A(h^{-1}|(x,y)-r(s)|)\log\left(\frac{1}{|(x,y)-r(s)|}\right) + B(h^{-1}|(x,y)-r(s)|) \tag{19}$$

where $A(z)$ and $B(z)$ are entire functions of $z^2 \in \mathbb{C}$ and each of them have elementary expressions in terms of Bessel functions (see [TZ] appendix A). $A(z)$ is the Riemann function [G].

We identify $(x,y) \in \mathbb{R}^2$ with $x+iy \in \mathbb{C}$, and introduce the notation $\rho(x+iy,r(t)) = \sqrt{(x+iy-r(t))\cdot(x-iy-\overline{r(t)})}$ where $z \mapsto \sqrt{z}$ is the positive square-root function with

$\sqrt{\operatorname{Re} z} > 0$ when $\operatorname{Re} z > 0$. Substitution of (19) in (17) implies that for $(x,y) \in \mathring{\Omega}$ and with $\partial_\nu := \partial_{\nu_s}$,

$$\varphi_h(x,y) = -\frac{1}{2}\int_{\partial\Omega} \varphi_h(r(s))\partial_\nu A(h^{-1}\rho)\log(\rho^2)\,dr(s)$$
$$-\frac{1}{2}\int_{\partial\Omega} \varphi_h(r(s))A(h^{-1}\rho)\partial_\nu\log(\rho^2)\,dr(s) + \int_{\partial\Omega}\partial_\nu B(h^{-1}\rho)\varphi_h(r(s))\,dr(s). \quad (20)$$

The holomorphic continuation of the third integral is the easiest to describe since there is a real analytic $F \in C^\omega(\mathbb{R},\mathbb{R})$ with entire extension $F^{\mathbb{C}} \in \mathcal{O}(\mathbb{C})$ satisfying

$$\partial_\nu B(h^{-1}\rho) = \partial_\nu\, F(h^{-2}\rho^2) \quad (21)$$

and the same is true for the normal derivative $\partial_\nu A(h^{-1}\rho)$ of the Riemann function. In view of (21), the last integral in (20) has a biholmorphic extension to $\Omega^{\mathbb{C}} := \{(z,w) \in \mathbb{C}^2; \operatorname{Re} z + i\operatorname{Re} w \in \Omega\}$.

In contrast, the first two integrals both turn out to have fairly subtle analytic continuations over Ω in $\mathbb{C}^2$ that rely heavily on analytic continuation of the eigenfunction boundary traces ([TZ] Appendix 9). However, we need only consider holomorphic continuation over a strictly interior curve $H \subset \mathring{\Omega}$ here. Thus, to describe the holomorphic continuation of the first integral on the right hand side of (20) it suffices to assume that $x+iy \in \mathring{\Omega}$ is far from the boundary with $|x+iy-r|^2 > 4\varepsilon_\circ^2 > 0$, where $\varepsilon_\circ < \operatorname{dist}(H,\partial\Omega)$. When $\max(|\operatorname{Im} w|, |\operatorname{Im} z|) < \varepsilon_\circ$, it follows by Taylor expansion that

$$|\rho^2(z+iw,r) - |\operatorname{Re} z + i\operatorname{Re} w - r|^2\,| \leq \max(|\operatorname{Im} w|, |\operatorname{Im} z|)|\operatorname{Re} z + i\operatorname{Re} w - r|.$$

Thus, $\operatorname{Re}\rho^2(z+iw,r) > \varepsilon_\circ^2$ and $(s, \operatorname{Re} z, \operatorname{Re} w) \mapsto \log(\rho^2(\operatorname{Re} z, \operatorname{Re} w, s))$ has a biholomorphic continuation in the $(\operatorname{Re} z, \operatorname{Re} w)$ variables to

$$[\Omega - \partial\Omega_{2\varepsilon_\circ}]^{\mathbb{C}}(\varepsilon_\circ) = \{(z,w) \in \Omega^{\mathbb{C}}; \min_{r\in\partial\Omega}|\operatorname{Re} z + i\operatorname{Re} w - r| \geq 2\varepsilon_\circ,\ |\operatorname{Im} z| \leq \varepsilon_\circ, |\operatorname{Im} w| \leq \varepsilon_\circ\}. \quad (22)$$

The same is true for $\partial_\nu A(\rho)$ and consequently, for the integral. By the same argument, the funcrtion $(s, \operatorname{Re} z, \operatorname{Re} w) \mapsto \partial_{\nu_s}\rho^2(\operatorname{Re} z, \operatorname{Re} w, s)$ also biholomorphically continues in the $(\operatorname{Re} z, \operatorname{Re} w)$-variables to $[\Omega - \partial\Omega_{2\varepsilon_\circ}]^{\mathbb{C}}(\varepsilon_\circ)$. Consequently, so does the second integral on the RHS of (20).

Restriction of the outgoing variables, (x,y) to $(q_1(s), q_2(s)) \in H$ in (20) yields the integral equation

$$N(h)\varphi_h^{\partial\Omega} = \varphi_h^H. \quad (23)$$

From now on, we will refer to $\varepsilon_\circ > 0$ as the *modulus of analyticity.* In light of the potential layer formula (23) for the Neumann eigenfunctions, it is useful to compare eigenfunction restrictions to $\partial\Omega$ with restrictions to $H \subset \mathring{\Omega}$ and similarly, for the holomorphic continuations. For the restrictions of the Neumann eigenfunctions pulled-back to the parameter domain, we continue to write

$$u_h^{\partial\Omega}(t) = \varphi_h^{\partial\Omega}(r_1(t), r_2(t)),\ \ u_h^H(t) = \varphi_h^H(q_1(t), q_2(t));\ \ t \in [-\pi,\pi] \quad (24)$$

with $r(t) = r_1(t) + ir_2(t) \in \partial\Omega$ and $q(t) = q_1(t) + iq_2(t) \in H$.

Proposition 2.1. *Suppose that $H \subset \Omega$ is real analytic and let dist$(H,\partial\Omega) = \min_{(s,t)\in[-\pi,\pi]^2}|q(t) - r(s)|$. Assume that $q(t)$ has a holomorphic continuation to $I(\delta) =$*

$[-\pi,\pi] \pm [-\delta,\delta]$. *Then the restriction* u_h^H *of the Neumann eigenfunctions has a holomorphic continuation* $u_h^{H,\mathbb{C}}(t)$ *to the strip* $S_{2\varepsilon_\circ,2\pi}$ *with*

$$2\varepsilon_\circ < \frac{dist(H,\partial\Omega)}{\sup_{t\in I_H(\delta)} |\partial_t q^{\mathbb{C}}(t)|}.$$

Moreover, in the strip $S_{2\varepsilon_\circ,2\pi}$, *the continuation is given by complexified potential layer equation*

$$N^{\mathbb{C}}(h)\varphi_h^{\partial\Omega} = \varphi_h^{H,\mathbb{C}}, \tag{25}$$

where $N^{\mathbb{C}}(h)$ *is the operator with Schwartz kernel* $N^{\mathbb{C}}(q^{\mathbb{C}}(t), r(s), h)$ *holomorphically continued in the outgoing t-variables, and* $\varphi_h^{H,\mathbb{C}}$ *the holomorphic continuation of* φ_h^H *to* $H_{\varepsilon_\circ}^{\mathbb{C}}$.

Proof. The proposition follows from the above analytic continuation argument for (20) and (23) since by (22) the u_h^H holomorphically continue to the set $\{t \in \mathbb{C}; \min_{r\in\partial\Omega} |q(\mathrm{Re}\, t) - r| \geq 2\varepsilon_\circ, |\mathrm{Im}\, q^{\mathbb{C}}(t)| < \varepsilon_\circ\}$. The formula in (25) follows from uniqueness of analytic continuation and the fact that, by the above analysis of (20), for $\zeta = q^{\mathbb{C}}(t) \in H_{2\varepsilon_\circ,2\pi}$,

$$\varphi_h^{H,\mathbb{C}}(\zeta) = [N(h)\varphi_h^{\partial\Omega}]^{\mathbb{C}}(\zeta) = N^{\mathbb{C}}(h)\varphi_h^{\partial\Omega}(\zeta).$$

□

3. The frequency function and measure of the nodal set

We first recall the definition of the frequency function with an important application due to F.H. Lin [L] for estimating measures of nodal sets. We are interested here in the planar case of holomorphic functions. In general, the frequency function for harmonic functions in arbitrary dimensions is defined as follows

Definition **3.1.** *Let* $\Delta u = 0$ *with* $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ *the standard Laplacian in* $\mathbb{R}^n$. *The frequency function of the harmonic function* u *in the unit ball* $B_1 \subset \mathbb{R}^n$ *is defined to be*

$$F(u) = \frac{\iint_{B_1} |\nabla u|^2}{\int_{\partial B_1} |u|^2}.$$

When the context is clear, we suppress the dependence of F on u and just write F for the frequency function. In the planar case, any non-zero holomorphic function $f(z)$ in the disc $B_1 = \{z \in \mathbb{C}; |z| \leq 1\}$, has a decomposition of the form $f = u + iv$ where u, v are harmonic conjugates and so, since $\partial_z f = \partial_x u + i\partial_x v$, in analogy with the harmonic case in Definition 3.1, one defines the frequency function to be

$$F = \frac{\iint_{B_1} |\partial_z f(z)|^2 dz d\bar{z}}{\int_{\partial B_1} |f(z)|^2 d\sigma(z)}. \tag{26}$$

An elementary but useful example to keep in mind is the monomial $f(z) = z^k = r^k e^{ki\theta}$; $k \in \mathbb{Z}^+$. In this case, one easily computes the frequency function to be $k^2 \int_0^1 r^{2k-1} dr = k/2$, where k is the degree of the polynomial z^k. By Green's formula, the analogous result is easily verified for arbitrary homogeneous harmonic polynomials in any dimension. The following result, proved by Lin [L] using Taylor expansion, and by Han [H] using Rouche's theorem is an important generalization of the polynomial case to arbitrary non-zero holomorphic

functions. We recall the result here and refer the reader to [H] for a proof (see also the upcoming book of Han and Lin [HL]). The key result that estimates the number of complex zeros of $f(z)$ in the disc B_1 is given by

THEOREM **3.2.** [H, L, HL] *Let $f(z)$ be a non-zero analytic function in $B_1 = \{z \in \mathbb{C} : |z| \leq 1\}$. Then, for some universal $\delta \in (0,1)$,*

$$\#\{f^{-1}(0) \cap B_\delta\} \leq 2F,$$

where F is defined to be the ratio in (26).

It is useful here to rewrite the frequency function F in (26) exclusively in terms of integrals over the circular disc boundary ∂B_1.

LEMMA **3.3.** *Let $f : B_1 \to \mathbb{C}$ be non-zero holomorphic. Then,*

$$F \leq \frac{\|\partial_\theta f\|_{L^2(\partial B_1)}}{\|f\|_{L^2(\partial B_1)}},$$

where $\partial_\theta = x\partial_y - y\partial_x$ is the unit tangential derivative along the circular boundary ∂B_1 of the disc.

Proof. The proof follows from Green's formula and an application of Cauchy-Schwarz. For $z = x+iy = (x,y) \in B_1$ we write $f(z) = \operatorname{Re} f(x,y) + i\operatorname{Im} f(x,y)$, where $\operatorname{Re} f(x,y), \operatorname{Im} f(x,y)$ are real-valued harmonic functions.

Since f is analytic, $\partial_z f = \partial_x \operatorname{Re} f - i\partial_y \operatorname{Re} f$, and so,

$$|\partial_z f|^2 = (\partial_x \operatorname{Re} f)^2 + (\partial_y \operatorname{Re} f)^2 = |\nabla(\operatorname{Re} f)|^2.$$

An application of Green's theorem implies that

$$\begin{aligned}
\iint_{B_1} |\partial_z f(z)|^2 dz d\overline{z} &= \iint_{B_1} |\nabla(\operatorname{Re} f)|^2 \, dxdy \\
&= \int_{\partial B_1} \operatorname{Re} f \cdot \partial_\nu(\operatorname{Re} f) \, d\theta - \iint_{B_1} \operatorname{Re} f \cdot \Delta(\operatorname{Re} f) dxdy \\
&= \int_{\partial B_1} \operatorname{Re} f \cdot \partial_\nu(\operatorname{Re} f) \, d\theta,
\end{aligned} \tag{27}$$

where, ν is the outward pointing unit normal to ∂B_1 and the last line follows since $\Delta(\operatorname{Re} f) = 0$ in B_1.

Next, we use the Cauchy-Riemann equations written in polar coordinates $(r,\theta) \in \mathbb{R}^+ \times [0, 2\pi)$ to rewrite the normal derivative term on the right hand side of the last line in (27) in terms of a tangential one.

$$\partial_\nu \operatorname{Re} f|_{\partial B_1} = \partial_r \operatorname{Re} f|_{r=1} = \partial_\theta \operatorname{Im} f|_{r=1}. \tag{28}$$

Hence, it follows from (28) and (27) that

$$\iint_{B_1} |\partial_z f(z)|^2 dz d\overline{z} = \int_{\partial B_1} \operatorname{Re} f \cdot \partial_\theta(\operatorname{Im} f) \, d\theta. \tag{29}$$

Finally, an application of Cauchy-Schwarz in (29) gives

$$\iint_{B_1} |\partial_z f(z)|^2 dz d\overline{z} \leq \|\operatorname{Re} f\|_{L^2(\partial B_1)} \cdot \|\partial_\theta (\operatorname{Im} f)\|_{L^2(\partial B_1)}$$
$$\leq \|f\|_{L^2(\partial B_1)} \cdot \|\partial_\theta f\|_{L^2(\partial B_1)}. \tag{30}$$

□

3.0.2. *Frequency functions for the holomorphic continuations of restricted eigenfunctions.* We wish to estimate here the intersection number $n(h,H)$ in terms of Lemma 3.3.

Proposition **3.4.** *Let $H \subset \mathring{\Omega}$ be a C^ω interior curve and $C_{\varepsilon_\circ}$ be a simply-connected, bounded domain in $\mathbb{C}$ containing the rectangle $S_{\varepsilon_\circ,\pi}$ with real-analytic boundary $\partial C_{\varepsilon_\circ}$ and arclength parametrization $t \mapsto \kappa(t) \in \partial C_{\varepsilon_\circ}$. Then, for $\varepsilon_\circ > 0$ sufficiently small*

$$n(h,H) \leq C_{H,\varepsilon_\circ} \frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}.$$

Here, $L^2_{\varepsilon_\circ} := L^2(\partial C_{\varepsilon_\circ}, |dt|)$ and ∂_T denotes the unit tangential derivative along $\partial C_{\varepsilon_\circ}$ with $\partial_T f(t) := \frac{d}{dt} f(\kappa(t))$.

Proof. Since $C_{\varepsilon_\circ}$ is a simply-connected bounded domain, by the Riemann mapping theorem there exists a conformal map

$$\kappa : \mathring{B}_1 \to C_{\varepsilon_\circ},$$

where $\mathring{B}_1 = \{z; |z| < 1\}$. By Caratheodory, there is $\tilde{\kappa} \in C^0(B_1)$ with $\tilde{\kappa}|_{\mathring{B}_1} = \kappa|_{\mathring{B}_1}$ univalent up to the boundary. Moreover, since $\partial C_{\varepsilon_\circ}$ is real-analytic, it follows from the Schwarz reflection principle that

$$\tilde{\kappa} \in C^\omega(B_1). \tag{31}$$

Analogous results also hold for the inverse conformal map $\kappa^{-1} : C_{\varepsilon_\circ} \to \mathring{B}_1$. Since κ is conformal and satisfies (31), it follows that the boundary restriction

$$\tilde{\kappa}|_{\partial B_1} : \partial B_1 \to \partial C_{\varepsilon_\circ}$$

is a C^ω-diffeomorphism. We define the composite function on B_1

$$g_h^{H,\mathbb{C}}(z) := u_h^{H,\mathbb{C}}(\tilde{\kappa}(z)); \;\; z \in B_1.$$

We apply theorem 3.2 to the holomorphic function $g_h^{H,\mathbb{C}}$ in B_1. We choose $\delta \in (0,1)$ so that $C_\delta := \tilde{\kappa}(B_\delta) \supset [-\pi,\pi]$. We have that

$$n(h,H) = N_{u_h} \cap [-\pi,\pi] \leq n^{\mathbb{C}}(h, C_\delta) = \#\{t \in C_\delta; u_h^{H,\mathbb{C}}(t) = 0\} = \#\{t \in B_\delta; g_h^{H,\mathbb{C}}(t) = 0\}. \tag{32}$$

It follows by Theorem 3.2, Lemma 3.3 and(32) that

$$n(h,H) \leq 2 \frac{\|\partial_\theta g_h^{H,\mathbb{C}}\|_{L^2(\partial B_1)}}{\|g_h^{H,\mathbb{C}}\|_{L^2(\partial B_1)}}. \tag{33}$$

An application of the change of variables formula in (33) with $t = \tilde{\kappa}(z)$ for $z \in \partial B_1$ proves the proposition. □

4. Estimating the frequency function: h-microlocal decomposition

In view of Proposition 3.4, we are left with showing that

$$\frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = \mathcal{O}_{\Omega,H}(h^{-1}). \tag{34}$$

To prove (34), we will need to h-microlocally decompose $\gamma_{\partial C_{\varepsilon_\circ}} u_h^{H,\mathbb{C}}$ where $\gamma_{\partial C_{\varepsilon_\circ}} : C^0(S_{2\varepsilon_\circ,2\pi}) \to C^0(\partial C_{\varepsilon_\circ})$ is the restriction map. We briefly digress here to introduce the relevant h-pseudodifferential cutoff operators noting that $\partial C_{\varepsilon_\circ}$ is C^ω-diffeomorphic to the unit circle ∂B_1.

4.1. **Semiclassical pseudodifferential operators on tori.**

Let M^n be compact manifold. The following semiclassical symbol spaces are standard [EZ] and will suffice for our purposes.

Definition **4.1.** *We say that $a \in S^{k,m}_{cl}(T^*M \times [0,h_0))$ if $a \in C^\infty(T^*M;[0,h_0))$ has an asymptotic expansion of the form $a \sim_{h\to 0^+} h^{-k}\sum_{j=0}^\infty a_j(x,\xi)h^j$ where*

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|};\ (x,\xi)\in T^*M.$$

The corresponding class of h-pseudodifferential operators $A_h : C^\infty(M) \to C^\infty(M)$ have Schwartz kernels locally of the form

$$A_h(x,y) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle/h} a(x,\xi;h)d\xi$$

with $a \in S^{k,m}_{cl}(T^*M;[0,h_0))$. We write $A_h = Op_h(a)$ for the operator with symbol $a(x,\xi;h)$.

Since $\partial C_{\varepsilon_\circ}$ is C^ω-diffeomorphic to a circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, it suffices here to consider h-pseudodifferential operators on tori and the latter operators can be conveniently described globally in terms of their action on Fourier coefficients. Given $A_h \in Op_h(S^{0,m}(T^*\mathbb{T}^n))$ one can write the Schwartz kernel in the form

$$A_h(x,y) = (2\pi)^{-n}\sum_{\xi\in(h\mathbb{Z})^n} e^{i\langle x-y,\xi\rangle/h} a_{\mathbf{T}^n}(x,\xi;h);\ (x,y)\in[-\pi,\pi]^n\times[-\pi,\pi]^n$$

where $a_{\mathbf{T}^n}(\cdot,\xi)\in C^\infty(\mathbf{T}^n)$ and

$$|\partial_x^\alpha \Delta_{h,\xi}^\beta a_{\mathbf{T}^n}(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

where $\Delta^\beta_{h,\xi} a_{\mathbf{T}^n}(x;\xi_1,...,\xi_n) = a_{\mathbf{T}^n}(x;\xi_1+h\beta_1,...,\xi_n+h\beta_n) - a_{\mathbf{T}^n}(x;\xi_1,...,\xi_n)$ is the semiclassical iterated difference operator in the frequency coordinates. The converse also holds, so that the two realizations of h-pseudodifferential operators are equivalent (see [Ag, Mc] for the homogeneous case where $h = 1$. The extension to the semiclassical setting is straightforward).

We are interested here specifically in the h-pseudodifferential cutoffs $\chi_h = Op_h(\chi) \in Op_h(S^{0,-\infty}(T^*\partial C_{\varepsilon_\circ}))$ where $\chi \in C_0^\infty(T^*\partial C_{\varepsilon_\circ})$. We naturally identify $\partial C_{\varepsilon_\circ}$ with $\mathbb{R}/2\pi\mathbb{Z}$ by using the periodic C^ω arclength parametrization

$$\kappa : [-\pi,\pi] \to \partial C_{\varepsilon_\circ};\ t \mapsto \kappa(t).$$

4.2. **Semiclassical wave front sets of eigenfunction restrictions.** Let $H^{n-1} \subset M^n$ be any interior *smooth* hypersurface in a compact manifold with or without boundary. In this subsection, we do not make any analyticity assumptions on either H or the ambient manifold, M. Let $u_h^H := \gamma_H \varphi_h$ be the eigenfunction restriction where $\gamma_H : f \mapsto f|_H, f \in C^0(H)$. Then, making a Fermi-coordinate decomposition in a collar neighbourhood of H, it is not hard to show that

$$WF_h(u_h^H) \subset B^*H = \{(s,\sigma) \in T^*H; |\sigma|_g \leq 1\}. \tag{35}$$

For Euclidean domains $M = \Omega$, (35) follows directly from potential layer formulas. For completeness and because of the importance of the localization of $WF_h(u_h^H)$ in our argument, we sketch the proof of (35) for planar domains, which is the case we are interested in here. The proof of (35) uses the potential layer representations of eigenfunctions discussed in Subsection 2.1.1 in the planar case $n = 2$ restricted to the curve H. It is immediate from (17) that

$$u_h^H(t) = \int_{-\pi}^{\pi} N(q(t), r(s); h)\; u_h^{\partial\Omega}(s) d\sigma(s). \tag{36}$$

Since $H \subset \Omega$ is interior, $\inf_{t,s \in [-\pi,\pi]} |q(t) - r(s)| \geq C > 0$ and so, from (16) it follows that

$$\tilde{N}(t,s;h) := N(q(t), r(s), h) = (2\pi h)^{-\frac{1}{2}} e^{ih^{-1}|q(t)-r(s)|} a(t,s;h) \tag{37}$$

where,

$$a(t,s;h) = \sum_{j=0}^{k} a_j(t,s) h^j + \mathcal{O}(h^{k+1})$$

uniformly for all $(q(t), r(s)) \in H \times \partial\Omega$ with $a_j \in C^\infty([-\pi,\pi] \times [-\pi,\pi])$. Similar uniform asymptotics hold for derivatives as well.

Let $\chi(\xi) \in C_0^\infty(\mathbb{R})$ be a cut-off function equal to zero when $|\xi| \geq 2$ and equal to 1 for $|\xi| < 3/2$ and let $Op_h(\chi) \in Op_h(S^{0,-\infty}(T^*H; (0, h_0]))$ be the microlocal cut-off with kernel

$$Op_h(\chi)(t,t') = (2\pi)^{-2} \sum_{\xi \in h\mathbb{Z}} e^{i\langle t-t', \xi\rangle/h}\, \chi(\xi); \;\; (t,t') \in [-\pi,\pi] \times [-\pi,\pi].$$

Then, from (36) and (37), it follows that

$$Op_h(1-\chi) u_h^H(t) = Op_h(1-\chi) N u_h^{\partial\Omega}(t)$$

$$= (2\pi)^{-2} \sum_{\xi \in h\mathbb{Z}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i[(t-t')\xi + |q(t') - r(s)|]/h}\, (1-\chi)(\xi)\, a(q(t'), r(s); h)\, u_h^{\partial\Omega}(s)\, ds\, dt.'$$

Since $|d_{t'} q(t')| = 1$, differentiation of the phase

$$\Psi(t,t',s;\xi) := (t-t')\xi + |q(t') - r(s)|$$

in t' gives

$$|\partial_{t'} \Psi(t,t',s;\xi)| = \Big| -\xi + \Big\langle d_{t'} q(t'), \frac{q(t') - r(s)}{|q(t') - r(s)|} \Big\rangle \Big| \geq |\xi| - 1 \geq \frac{1}{2}; \text{ when } |\xi| \geq \frac{3}{2}.$$

Since $|\xi| \geq \frac{3}{2}$ when $\xi \in \operatorname{supp}\chi$, repeated integration by parts in t', an application of Cauchy-Schwarz and using that $\|u_h^{\partial\Omega}\|_{L^2} = \mathcal{O}(h^{-1/4})$ [BGT] implies that $\sup_{t \in [0,2\pi]} |Op_h(1-$

$\chi(\xi))u_h^H(t)| = \mathcal{O}(h^\infty \langle\xi\rangle^{-\infty})$ where $\langle\xi\rangle := \sqrt{1+|\xi|^2}$. The same argument for t-derivatives combined with the Sobolev lemma implies that for all $k \in \mathbb{Z}^+$,

$$\|Op_h(1-\chi(\xi))u_h^H\|_{C^k([-\pi,\pi])} = \mathcal{O}_k(h^\infty \langle\xi\rangle^{-\infty}). \tag{38}$$

The wavefront bound in (35) is an immediate consequence of (38) since the cutoff function $\chi(\xi)$ can be chosen with support arbitrarily close to $|\xi| = 1$ and the same argument gives (38) for any such cutoff.

In the next section we improve the compactness result (35) under the real-analyticity assumption on $(\partial\Omega, H)$ to show that in the h-microlocal decomposition (5) the residual term $\|\partial_T(1-Op_h(\chi_R))u_h^{H,\mathbb{C}}\|_{\varepsilon_\circ}^2 = \mathcal{O}(e^{-C_0\langle R\rangle/h})$ with appropriate $C_0 > 0$ and where $\chi_R \in C_0^\infty(\mathbb{R})$ with supp $\chi_R \subset \{\xi; |\xi| \leq R\}$. Hence, to get an asymptotic estimate for the frequency function of $u_h^{H,\mathbb{C}}$, it suffices to bound $\|\partial_T Op_h(\chi_R)u_h^{H,\mathbb{C}}\|_{\varepsilon_\circ}$ and the latter is $\mathcal{O}(h^{-1}\|u^{H,\mathbb{C}}\|_{\varepsilon_\circ})$ by standard L^2-boundedness of the h-pseudodifferential operator $h\partial_T\chi_h \in Op_h(S^{0,-\infty}(T^*\partial C_{\varepsilon_\circ}))$.

4.3. **The real analytic case.** We now assume that H is real-analytic. As outlined in the previous section, our goal here is to improve the $\mathcal{O}(h^\infty)$-bound in (38) to obtain exponential decay estimates for the residual mass term of the form $\|Op_h(1-\chi)u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} = \mathcal{O}(e^{-C_0/h})$. In the following, using the parametrization $[-\pi,\pi] \ni t \mapsto \kappa(t)$, we identify $\partial C_{\varepsilon_\circ}$ with $\mathbb{R}/(2\pi\mathbb{Z})$ and so, $Op_h(1-\chi) : C^\infty(\mathbb{R}/2\pi\mathbb{Z}) \to C^\infty(\mathbb{R}/2\pi\mathbb{Z})$.

4.3.1. *Holomorphic continuation of the $\tilde{N}(t,s;h)$-kernel.* Given $(z,w) \in \mathbb{C}^2$, consider the map $z+iw \mapsto (z+iw)^* = z - iw$ which is the holomorphic continuation to $\mathbb{C}^2$ of the usual complex conjugation $x+iy \mapsto x-iy$ when $(x,y) \in \mathbb{R}^2$. In the following, $z \mapsto z^{1/2}$ denotes the positive square root with $-\pi < arg(z) \leq \pi$.

In view of Proposition 2.1 it follows that for $\varepsilon_\circ > 0$ sufficiently small, the potential layer equation $u_h^H(t) = Nu_h^{\partial\Omega}(t)$ analytically continues to the equation

$$u_h^{H,\mathbb{C}}(\zeta) = [Nu_h^{\partial\Omega}]^{\mathbb{C}}(\zeta);\ \ \zeta \in S_{2\varepsilon_\circ,2\pi}. \tag{39}$$

In particular, we consider here the case where $\zeta = \kappa(t) \in \partial C_{\varepsilon_\circ}$.

For $\zeta \in U_{\varepsilon_\circ}$, where $U_{\varepsilon_\circ} := \{\zeta \in S_{2\varepsilon_\circ,2\pi};\ \max_{z\in\partial C_{\varepsilon_\circ}} |z-\zeta| < \frac{\varepsilon_\circ}{2}\}$, equation (39) remains valid and moreover, since

$$\text{Re } [q^{\mathbb{C}}(\zeta) - r(s)][q^{\mathbb{C}}(\zeta)^* - \overline{r(s)}] \gtrapprox \varepsilon_\circ^2 > 0 \text{ when } (\zeta,s) \in U_{\varepsilon_\circ} \times [-\pi,\pi], \tag{40}$$

the kernel

$$N^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s), h) = \text{Ha}_1^{(1)}\left(h^{-1}\sqrt{[q^{\mathbb{C}}(\zeta) - r(s))][q^{\mathbb{C}}(\zeta)^* - \overline{r(s)})]}\right) \tag{41}$$

is holomorphic for $\zeta \in U_{\varepsilon_\circ}$. By Proposition 2.1, we have

$$u_h^{H,\mathbb{C}}(\zeta) = \int_{-\pi}^{\pi} N^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s), h)\, u_h^{\partial\Omega}(s) d\sigma(s),\ \ \zeta \in U_{\varepsilon_\circ}. \tag{42}$$

It follows from (40), (41) and the integral formula (18) that the real WKB asymptotics for the $N(t,s,h)$-kernel [HZ, TZ] holomorphically continues in t to give the complex asymptotic formula

$$N^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s), h) = (2\pi h)^{-\frac{1}{2}} e^{i\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s))/h} a^{\mathbb{C}}(\zeta, s; h);\ \ (\zeta,s) \in U_{\varepsilon_\circ} \times [-\pi,\pi], \tag{43}$$

where, $a^{\mathbb{C}}(\zeta,s;h) \sim_{h\to 0} \sum_{k=0}^{\infty} a_k^{\mathbb{C}}(\zeta,s)h^k$ with $a_k(\cdot,s) \in \mathcal{O}(U_{\varepsilon_\circ})$ and

$$\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s)) = \sqrt{[q^{\mathbb{C}}(\zeta) - r(s)][q^{\mathbb{C}}(\zeta)^* - \overline{r(s)}]};\ (\zeta,s) \in U_{\varepsilon_\circ} \times [-\pi,\pi].$$

In particular, for $\zeta = \kappa(t) \in \partial C_{\varepsilon_\circ}$, we have

$$u_h^{H,\mathbb{C}}(\kappa(t)) = \int_{-\pi}^{\pi} N^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t)), r(s), h)\, u_h^{\partial\Omega}(s) d\sigma(s),\ \ t \in [-\pi,\pi], \tag{44}$$

where $N^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t)), r(s), h)$ satisfies the asymptotics in (43). Since we compute in the parametrization variables $(t,s) \in [-\pi,\pi]$, to simplify notation we define

$$\tilde{N}^{\mathbb{C}}(t,s,h) := N^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t)), r(s), h);\ \ (t,s) \in [-\pi,\pi] \times [-\pi,\pi]. \tag{45}$$

4.3.2. *Estimating the residual kernel* $[Op_h(1-\chi)\tilde{N}^{\mathbb{C}}](t,s;h)$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff with $\chi(\xi) = 0$ when

$$|\xi| \geq 20\varepsilon_\circ^{-1} \sup_{(\zeta,s)\in U_{\varepsilon_\circ}\times[-\pi,\pi]} |\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s))|$$

and $\chi(\xi) = 1$ when

$$|\xi| \leq 10\varepsilon_\circ^{-1} \sup_{(\zeta,s)\in U_{\varepsilon_\circ}\times[-\pi,\pi]} |\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s))|.$$

In this section we prove

PROPOSITION **4.2.** *Let $H \subset \Omega$ be C^ω interior curve with* $\operatorname{dist}(H,\partial\Omega) < \delta(\varepsilon_\circ)$ *and $\partial C_{\varepsilon_\circ}$ be a curve satisfying (15). Then, assuming $\delta(\varepsilon_0) > 0$ is sufficiently small and $k \in \mathbb{Z}^+$, there is a constant $C_k(\varepsilon_\circ) > 0$ such that for $h \in (0, h(\varepsilon_\circ)]$,*

$$\|\, [Op_h(1-\chi)\tilde{N}^{\mathbb{C}}](\cdot,\cdot;h)\, \|_{C^k([-\pi,\pi]\times[-\pi,\pi])} = \mathcal{O}(e^{-C_k(\varepsilon_\circ)/h}).$$

Proof. In light of the complexified potential layer formula in (42), we substitute the complex WKB asymptotics for $N^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s), h)$ in (43) and use the Cauchy integral formula to deform contours of integration.

From (42) and (43), one gets that

$$\begin{aligned} &[Op_h(1-\chi)\tilde{N}^{\mathbb{C}}](t,s,h) \\ &= (2\pi)^{-2} \sum_{\xi\in h\mathbb{Z}} \int_{-\pi}^{\pi} e^{i[(t-t')\xi + \rho^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t')), r(s))]/h}\, (1-\chi)(\xi)\, a^{\mathbb{C}}(\kappa(t'), r(s); h)\, dt'. \end{aligned} \tag{46}$$

Consider the complex phase

$$\Psi^{\mathbb{C}}(t,t',s) := (t-t')\xi + \rho^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t')), r(s)).$$

For simplicity, write $\rho^{\mathbb{C}}(t',s)$ for $\rho^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(t')), r(s))$. Consider for $\xi \in h\mathbb{Z}$ the deformed contour

$$\omega_\xi(t') = t' - i\frac{\varepsilon_\circ}{2}\operatorname{sgn}(\xi). \tag{47}$$

where $(t,t',s) \in [-\pi,\pi]^3$. The deformed phase function

$$\Psi(t,\omega_\xi(t'),s) = \Psi\left(t, t' - i\frac{\varepsilon_\circ}{2}\operatorname{sgn}(\xi), s\right) = (t-t')\xi + i\frac{\varepsilon_\circ}{2}|\xi| + \rho^{\mathbb{C}}(\omega_\xi(t'), s). \tag{48}$$

Since $|\xi| \geq 10\varepsilon_\circ^{-1} \sup_{(\zeta,s)\in U_{\varepsilon_\circ}\times[-\pi,\pi]} |\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s))|$ when $\xi \in \operatorname{supp}(1-\chi)$, it follows from (48) that

$$\operatorname{Im}\Psi(t, \omega_\xi(t'), s) \geq 4 \sup_{(\zeta,s)\in U_{\varepsilon_\circ}\times[-\pi,\pi]} |\rho^{\mathbb{C}}(q^{\mathbb{C}}(\zeta), r(s))| \gtrapprox \varepsilon_\circ \tag{49}$$

uniformly for $(t,t',s) \in [-\pi,\pi]^3$. Moreover, for $|\xi| \gg 1$ it also follows from (48) that

$$\operatorname{Im}\Psi(t, \omega_\xi(t'), s) = \frac{\varepsilon_\circ}{2}|\xi| + \mathcal{O}(1) \geq \frac{\varepsilon_\circ}{3}|\xi|. \tag{50}$$

Using Cauchy's theorem, we deform the t'-contour of integration in (46) to get

$$\begin{aligned} &[Op_h(1-\chi)\tilde{N}^{\mathbb{C}}](t,s,h) \\ &= (2\pi)^{-2} \textstyle\sum_{\xi\in h\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\Psi(t,\omega_\xi(t'),s;\xi)/h} \, (1-\chi)(\xi) \, a^{\mathbb{C}}(\kappa^{\mathbb{C}}(\omega_\xi(t')), r(s); h) \, dt' \end{aligned} \tag{51}$$

where the imaginary part of the deformed phase function $\Psi(t, \omega_\xi(t'), s)$ satisfies (49). It follows from (49) and (50) that for appropriate $C(\varepsilon_\circ) \gtrapprox \varepsilon_\circ$,

$$|[Op_h(1-\chi)\tilde{N}^{\mathbb{C}}(t,s,h)| \leq e^{-\frac{C(\varepsilon_\circ)}{h}} \times \left(\sum_{|\xi|\geq 1} e^{-\frac{\varepsilon_\circ}{4h}|\xi|} \right) = \mathcal{O}(e^{-\frac{C(\varepsilon_\circ)}{h}}). \tag{52}$$

The argument for the higher C^k-norms is basically the same since the complex phase function $\Psi^{\mathbb{C}}(t,t',s)$ is unchanged. The derivatives ∂_s^α and ∂_t^β just create additional polynomial powers in h^{-1} in the amplitude $a^{\mathbb{C}}(\cdot,\cdot;h)$.

□

Remark: For future reference (see proof of Theorem 1.1 below), we note that when $\chi_R \in C_0^\infty(\mathbb{R})$ with $\chi_R(\xi) = 1$ for $|\xi| < R$ and $\operatorname{supp} \chi_R \subset \{\xi; |\xi| < 2R\}$, it is clear from (50) that

$$\|Op_h(1-\chi_R)\tilde{N}^{\mathbb{C}}(\cdot,\cdot,h)\|_{C^k} = \mathcal{O}_k(e^{-\frac{C_R(\varepsilon_\circ)}{h}}), \tag{53}$$

where $C_R(\varepsilon_\circ) \gtrapprox R$ as $R \to \infty$.

5. Proof of Theorem 1.1

Proof. Let $\chi_R \in C_0^\infty(\mathbb{R};[0,1])$ be a frequency cutoff as in Proposition 4.2 with $\chi_R(\xi) = 1$ for $|\xi| \leq R$ and $\chi_R(\xi) = 0$ for $|\xi| \geq 2R$. To simplify notation, in the following we continue to write $L^2_{\varepsilon_\circ} = L^2(\partial C_{\varepsilon_\circ})$ (resp. $L^2 = L^2([-\pi,\pi])$) and the corresponding unit speed parameterizations are $t \mapsto \kappa(t)$ (resp. $t \mapsto q(t)$).

We recall that the basic frequency function estimate gives

$$\begin{aligned} n(h,H) &\leq h^{-1} \frac{\|h\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} \\ &\leq h^{-1} \Big(\frac{\|Op_h(\chi_R)(h\partial_T) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} + \frac{\|(1-Op_h(\chi_R))(h\partial_T) u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_{h,H}^{\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} \Big). \end{aligned}$$

From Proposition 4.2 and Cauchy-Schwarz, it follows that

$$\frac{\|(1-Op_h(\chi_R))h\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = \mathcal{O}\left(\frac{e^{-\frac{C_R(\varepsilon_\circ)}{h}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}\right). \tag{54}$$

In the last line of (54) we have used that $\|u_h^{\partial\Omega}\|_{L^2} = \mathcal{O}(h^{-\alpha})$ for $\alpha > 0$ (for example, Tataru's sharp bound gives $\alpha = 1/3$). Since $u_h^{H,\mathbb{C}}(t)$ is holomorphic for all $t \in S_{2\varepsilon_\circ,2\pi}$, it follows from the Cauchy integral formula (see figure 1) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\sup_{t\in S_{\varepsilon_\circ,\pi}} |u_h^{H,\mathbb{C}}(t)| &\leq C_2 \sup_{t\in S_{\varepsilon_\circ,\pi}} \cdot\frac{1}{4\pi^2}\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\kappa(s)-t|^{-2}\,dsdt\right)^{\frac{1}{2}} \cdot \|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}} \\ &= \mathcal{O}(1)\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}.\end{aligned} \tag{55}$$

In (55) we use that $\partial C_{\varepsilon_\circ}$ and $[-\pi,\pi]$ are disjoint so that $f(s,t) = |\kappa(s)-t|^{-1} \in L^2([-\pi,\pi]\times[-\pi,\pi])$. Substitution of (55) in (54) then implies that

$$\frac{\|(1-Op_h(\chi_R))h\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = \mathcal{O}\left(e^{\frac{-C_R(\varepsilon_\circ)}{h}}\|u_h^{H,\mathbb{C}}\|^{-1}_{L^\infty(S_{\varepsilon_\circ,\pi})}\right) = \mathcal{O}(e^{\frac{-C_R(\varepsilon_\circ)+C_0}{h}}), \tag{56}$$

since by assumption $\|u_h^{H,\mathbb{C}}\|_{L^\infty(S_{\varepsilon_\circ,\pi})} \geq e^{-\frac{C_0}{h}}$ for some $C_0 > 0$. Since $Op_h(\chi_R)(h\partial_T) \in Op_h(S^{0,-\infty}(T^*H))$, it follows by L^2-boundedness that

$$\frac{\|Op_h(\chi_R)h\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\varepsilon_\circ}}} = \mathcal{O}_{R,\varepsilon_\circ}(1). \tag{57}$$

The constant $C_R(\varepsilon_\circ) \gtrapprox R$ as $R \to \infty$, and so, the proof of Theorem 1.1 follows from (56) and (57), by choosing R sufficiently large so that $C_R(\varepsilon_\circ) - C_0 > 0$ in (56). ☐

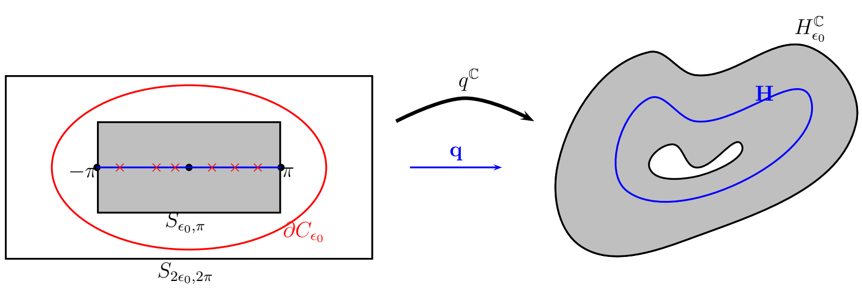

FIGURE 1.

6. Proof of Theorem 1.2

Proof. To prove Theorem 1.2, we will need the following operator bound:

Proposition **6.1.** *Assume that Ω is a smooth convex bounded domain and the interior curve H is strictly convex and let $\chi_{\varepsilon_\circ} \in C_0^\infty(S_{\varepsilon_\circ,\pi})$ with $\chi_{\varepsilon_\circ}(t) = 1$ when $\frac{\varepsilon_\circ}{3} \leq \operatorname{Im} t \leq \frac{2\varepsilon_\circ}{3}$ and $\chi_{\varepsilon_\circ}(t) = 0$ when $\operatorname{Im} t \geq \frac{5\varepsilon_\circ}{6}$ and $\operatorname{Im} t \leq \frac{\varepsilon_\circ}{6}$. Then, for $\varepsilon_\circ > 0$ sufficiently small, there is a plurisubharmonic weight function $\rho \in C^\omega(supp\, \chi_{\varepsilon_\circ}; \mathbb{R}^+)$ and operators $P(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ with decomposition*

$$P(h) = h^{\frac{1}{2}} Op_h(\chi_{\mathcal{G}}) + R(h),$$

where $\chi_{\mathcal{G}} \in C_0^\infty(B^*\partial\Omega)$ and

$$\int\int_{S_{\varepsilon_\circ,\pi}} e^{-2\frac{\rho(t)}{h}} |u_h^{H,\mathbb{C}}(t)|^2 \, \chi_{\varepsilon_\circ}(t) \, dt d\bar{t} = \langle P(h)\varphi_h^{\partial\Omega}, \varphi_h^{\partial\Omega} \rangle_{L^2}. \tag{58}$$

Moreover,

$$\int_{B^*\partial\Omega} \chi_{\mathcal{G}}(y,\eta) \, dy d\eta \geq C_0(\Omega, H, \varepsilon_\circ) > 0$$

and the remainder operator $R(h)$ satisfies $\|R(h)\|_{L^2 \to L^2} = \mathcal{O}(h)$.

The proof of Proposition 6.1 is rather technical and to avoid breaking the exposition at this point, we defer the proof to section 8. Assuming Proposition 6.1 for the moment, as discussed in subsection 1.2, the proof of Theorem 1.2 then follows readily from Theorem 1.1 and Proposition 6.1. Indeed, from the Rellich commutator argument in [Bu] it follows that quantum ergodicity of the interior eigenfunctions φ_h implies that the boundary restrictions $\varphi_h^{\partial\Omega}$ also have the analogous quantum ergodic restriction property in (13). We note that the last statement is not necessarily true if one replaces $\partial\Omega$ by an arbitrary interior curve, H. □

7. Asymptotics for the complexified potential layer operator $N^{\mathbb{C}}(h)$

To simplify the writing somewhat, we assume throughout this section that $\partial\Omega$ is smooth. The case of boundaries with corners is discussed in the final subsection 11.

Abusing notation somewhat we let $\rho^{\mathbb{C}}(t,s) = \rho^{\mathbb{C}}(q^{\mathbb{C}}(t), r(s))$ for $(t,s) \in S_{2\varepsilon_\circ,2\pi} \times [-\pi,\pi]$ (see (43)) and define the weight function

$$S(t) := \max_{s \in [-\pi,\pi]} \operatorname{Re}\, [i\rho^{\mathbb{C}}(t,s)], \tag{59}$$

one has the following

Lemma **7.1.** *For $q^{\mathbb{C}}(t) \in H_{\varepsilon_\circ}^{\mathbb{C}}$ and with the weight function $S(t)$ in (59), there exist $b_j^{\mathbb{C}}(\cdot,s) \in \mathcal{O}(S_{2\varepsilon_\circ,2\pi}; C^\omega(\mathbb{R}/2\pi\mathbb{Z})); j \geq 0$ such that*

$$e^{-S(t)/h} \cdot N^{\mathbb{C}}(q^{\mathbb{C}}(t), r(s); h) = (2\pi h)^{-1/2} \exp\left([i\rho^{\mathbb{C}}(t,s) - S(t)]/h\right) \left(\sum_{j=0}^{N} b_j^{\mathbb{C}}(t,s) h^j \right) + \mathcal{O}(h^{N+1}). \tag{60}$$

Proof. The lemma is an immediate consequence of Proposition 2.1 and (43) since

$$-S(t) + \operatorname{Re}(i\rho^{\mathbb{C}}(t,s)) \leq 0, \;\; (t,s) \in S_{2\varepsilon_\circ,2\pi} \times [-\pi,\pi].$$

□

The main step in the proof of Proposition 6.1 is an analysis of the asymptotics of the composite operators $P(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$, where

$$P(h) = [e^{-S/h}\chi_{\varepsilon_\circ} N^{\mathbb{C}}(h)]^* \cdot [e^{-S/h}\chi_{\varepsilon_\circ} N^{\mathbb{C}}(h)].$$

For this, one needs a detailed analysis of the complex phase function on the right hand side of (60). We begin with

7.1. **Asymptotic expansion of $\rho^{\mathbb{C}}(t,s)$.** Let $T_H(s) = d_s q(s)$ be the unit tangent to H and $\nu_H(s)$ the unit outward normal to H. Throughout the paper, $\kappa_H(s)$ denotes the scalar curvature of H. In the following, it will be useful to define the relative displacement vector

$$d(s, \mathrm{Re}\, t) := \frac{q(\mathrm{Re}\, t) - r(s)}{|q(\mathrm{Re}\, t) - r(s)|}.$$

From the Frenet-Serret formulas, we get that for $\delta > 0$ small, the holomorphic continuation $q^{\mathbb{C}}$ of the parametrization q of H satisfies for $|\mathrm{Im}\, t| \le \delta$,

$$\begin{aligned} q^{\mathbb{C}}(\mathrm{Re}\, t + i\mathrm{Im}\, t) - r(s) &= q(\mathrm{Re}\, t) - r(s) + i\mathrm{Im}\, t\, T_H(\mathrm{Re}\, t) - \tfrac{1}{2}\kappa_H(\mathrm{Re}\, t)|\mathrm{Im}\, t|^2 \nu_H(\mathrm{Re}\, t) \\ &-\tfrac{i}{6}(\mathrm{Im}\, t)^3[\kappa_H'(\mathrm{Re}\, t)\nu_H(\mathrm{Re}\, t) - \kappa_H^2(\mathrm{Re}\, t) T_H(\mathrm{Re}\, t)] + \mathcal{O}(|\mathrm{Im}\, t|^4). \end{aligned} \tag{61}$$

Similarly, when $|t - s| \le \delta$, one also has the expansion

$$\begin{aligned} q^{\mathbb{C}}(\mathrm{Re}\, t + i\mathrm{Im}\, t) - q(s) &= (\mathrm{Re}\, t + i\mathrm{Im}\, t - s)T_H(s) + \tfrac{1}{2}\kappa_H(s)(\mathrm{Re}\, t + i\mathrm{Im}\, t - s)^2\nu_H(s) \\ &+\tfrac{1}{6}[\kappa_H'(s)\nu_H(s) - \kappa_H^2(s)T_H(s)]\,(\mathrm{Re}\, t + i\mathrm{Im}\, t - s)^3 + \mathcal{O}(|\mathrm{Re}\, t + i\mathrm{Im}\, t - s|^4). \end{aligned} \tag{62}$$

Both (61) and (62) will be useful at different points in our analysis; the former when determining growth of functions in $\mathrm{Im}\, t$ and the latter when estimating joint growth in $\mathrm{Re}\, t - s$ and $\mathrm{Im}\, t$.

Let $\langle , \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the standard complex bilinear extension of the Cartesian inner product on $\mathbb{R} \times \mathbb{R}$. A direct computation using (61) gives

$$\begin{aligned} \mathrm{Im}\, \rho^{\mathbb{C}}(t,s) &= \langle d(s, \mathrm{Re}\, t), T_H(\mathrm{Re}\, t)\rangle\,(\mathrm{Im}\, t) - \Big(\tfrac{1}{6}\langle \kappa_H'(\mathrm{Re}\, t)\nu_H(\mathrm{Re}\, t) - \kappa_H^2(\mathrm{Re}\, t)T_H(\mathrm{Re}\, t), d(s,\mathrm{Re}\, t)\rangle \\ &-\tfrac{1}{2}\kappa_H(\mathrm{Re}\, t)\langle \nu_H(\mathrm{Re}\, t), d(s, \mathrm{Re}\, t)\rangle + \tfrac{1}{2}\langle d(s,\mathrm{Re}\, t), T_H(\mathrm{Re}\, t)\rangle\, |q(\mathrm{Re}\, t) - r(s)|^{-2} \\ &-\tfrac{1}{2}|q(\mathrm{Re}\, t) - r(s)|^{-2}\langle d(s,\mathrm{Re}\, t), T_H(\mathrm{Re}\, t)\rangle^3\Big)\,(\mathrm{Im}\, t)^3 + \mathcal{O}(|\mathrm{Im}\, t|^5). \end{aligned} \tag{63}$$

It follows that at a critical point $s = s^*(t)$ of $\mathrm{Im}\, \rho^{\mathbb{C}}(t,s)$

$$\partial_s \mathrm{Im}\, \rho^{\mathbb{C}}(t, s^*(t)) = 0, \tag{64}$$

and when $\mathrm{Im}\, t \neq 0$, we have

$$\langle \partial_s d(s^*(t), \mathrm{Re}\, t), T_H(\mathrm{Re}\, t)\rangle + \mathcal{O}(|\mathrm{Im}\, t|^2) = 0. \tag{65}$$

Moreover, when equation (65) is satisfied, we have

Lemma **7.2.** *Let $t \in [-\pi, \pi] + i[\frac{\delta}{2}, \delta]$ solve the critical point equation in (65). Then, for $\delta > 0$ sufficiently small,*

$$|\langle T_H(\mathrm{Re}\, t), d(s^*(t), \mathrm{Re}\, t)\rangle| = 1 + \mathcal{O}(|\mathrm{Im}\, t|^2).$$

Proof. Carrying out the s-differentiation gives

$$\langle \partial_s d(s, \operatorname{Re} t), T_H(\operatorname{Re} t)\rangle = |q(\operatorname{Re} t) - r(s)|^{-1}\Big(\langle T_{\partial\Omega}(s), T_H(\operatorname{Re} t)\rangle \\ -\langle T_{\partial\Omega}(s), d(s, \operatorname{Re} t)\rangle \cdot \langle T_H(\operatorname{Re} t), d(s, \operatorname{Re} t)\rangle\Big), \tag{66}$$

where $T_{\partial\Omega}(s) = d_s r(s)$ the unit tangent to $\partial\Omega$.
Since $|T_H(\operatorname{Re} t)| = |T_{\partial\Omega}(s)| = |d(s, \operatorname{Re} t)| = 1$, it follows from (66) and the cosine law $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$ that $\partial_s\langle d, T_H\rangle = 0$ if and only if either

$$(i)\ \ |\langle T_{\partial\Omega}(s), d(s, \operatorname{Re} t)\rangle| = 1$$

or

$$(ii)\ \ |\langle T_H(\operatorname{Re} t), d(s, \operatorname{Re} t)\rangle| = 1.$$

The identity (i) is never satisfied since H is by assumption an interior curve and $\partial\Omega$ is convex, so it is supported by the tangent line at each point of the boundary. As a result, (ii) must hold and this finishes the proof. □

Given, its geometric significance, in view of Lemma 7.2 it makes sense to single the points $y = s(\operatorname{Re} t)$ which solve the approximate critical point equation

$$\langle T_H(\operatorname{Re} t), d(s(\operatorname{Re} t), \operatorname{Re} t)\rangle = -1. \tag{67}$$

Geometrically, $q(s(\operatorname{Re} t)) \in \partial\Omega$ is the boundary intersection of the billiard trajectory in Ω that tangentially glances $H \subset \Omega$ at $q(\operatorname{Re} t)$. By convexity there are two such points on the boundary and the condition $\langle d(s, \operatorname{Re} t), T_H(\operatorname{Re} t)\rangle = -1$ uniquely specifies the point.

Remark: In the next section, we improve the result in Lemma 7.2 and show that in fact $|\langle T_H(\operatorname{Re} t), d(s(\operatorname{Re} t), \operatorname{Re} t)\rangle| = 1+\mathcal{O}(|\operatorname{Im} t|^4)$ which implies that the holomorphic continuation $s(t)$ of the geometric solution of (67) agrees to $\mathcal{O}(|\operatorname{Im} t|^5)$-error with the exact critical point $s^*(t)$ in (65). We then use this fact to determine the asymptotics of the weight $S(t)$ to $\mathcal{O}(|\operatorname{Im} t|^5)$-accuracy.

7.2. **Glancing sets relative to H.** We start by defining the glancing set relative to H. The real part of complex phase $\operatorname{Re} i\rho^{\mathbb{C}}(t, s)$ attains a maximum at $s = s(t)$ and as we show in (74) below, modulo $\mathcal{O}(|\operatorname{Im} t|^5)$-error terms, the weight function $S(t)$ equals $\operatorname{Re} i\rho^{\mathbb{C}}(t, s(t))$. The points $s(t)$ have a simple geometric characterization in terms of glancing sets relative to H, which we now describe. Unless specified otherwise, when t is complex we assume here that $\operatorname{Im} t \geq 0$.

Lemma **7.3.** *For fixed $s \in [-\pi, \pi]$ let $y(s)$ be a solution of $T_H(\cdot) = -d(s, \cdot)$. Then, the map $y : [-\pi, \pi] \to [-\pi, \pi]$ defined by $s \mapsto y(s)$ induces a real-analytic diffeomorphism of $\mathbb{R}/2\pi\mathbb{Z}$ (which we also denote by y). The inverse map will be denoted by $y = s(\operatorname{Re} t)$.*

Proof. The equation $|\langle T_H(\operatorname{Re} t), d(s, \operatorname{Re} t)\rangle| = 1$ is equivalent to

$$\langle \nu_H(\operatorname{Re} t), q(\operatorname{Re} t) - r(s)\rangle = 0 \ \ (*).$$

Unlike the defining equation in (67), $(*)$ has the advantage of being non-degenerate in $\operatorname{Re} t$. Indeed, differentiating the left hand side of $(*)$ with respect to $\operatorname{Re} t$ yields

$$\kappa_H(\operatorname{Re} t)\langle T_H(\operatorname{Re} t), q(\operatorname{Re} t) - r(s)\rangle + \langle \nu_H(\operatorname{Re} t), T_H(\operatorname{Re} t)\rangle = \kappa_H(\operatorname{Re} t)\langle T_H(\operatorname{Re} t), q(\operatorname{Re} t) - r(s)\rangle$$

Evaluating the last expression on the right hand side at $\mathrm{Re}\, t = y(s)$, implies that

$$
\begin{aligned}
&|\, \partial_{\mathrm{Re}\, t} \langle \nu_H(\mathrm{Re}\, t), q(\mathrm{Re}\, t) - r(s) \rangle \,|\, |_{\mathrm{Re}\, t = y(s)} \\
&= \kappa_H(y(s))\, |q(y(s)) - r(s)| \geq \min_{p \in H} \kappa_H(p) \cdot dist(H, \partial\Omega) > 0,
\end{aligned}
\tag{68}
$$

given that $\kappa_H > 0$. From (68), the implicit function theorem gives two analytic solution curves $\mathrm{Re}\, t \mapsto s_\pm(\mathrm{Re}\, t)$ solving $\langle T_H(\mathrm{Re}\, t), d(s_\pm(\mathrm{Re}\, t), \mathrm{Re}\, t\rangle = \pm 1$. Similarily,

$$
\begin{aligned}
&|\, \partial_s \langle \nu_H(\mathrm{Re}\, t), q(\mathrm{Re}\, t) - r(s) \rangle \,|\, |_{\mathrm{Re}\, t = y(s)} \\
&= |\langle \nu_H(\mathrm{Re}\, t), T_{\partial\Omega}(s) \rangle| \geq C(H, \partial\Omega) > 0,
\end{aligned}
\tag{69}
$$

since H is interior and Ω is convex. Thus, there are two smooth solution curves $s \mapsto y_\pm(s)$ solving $\langle T_H(y_\pm(s)), d(s, y_\pm(s)\rangle = \pm 1$. We choose here $y(s) = y_-(s)$. In the case where $\mathrm{Im}\, t < 0$, one chooses $y(s) = y_+(s)$. The mapping $y : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ is clearly bijective due to the positive curvature of H. □

DEFINITION **7.4.** *We define the* glancing set relative to H *for the billiard flow in* Ω *to be the set*

$$\Sigma := \{(r(s), q(\mathrm{Re}\, t)) \in \partial\Omega \times H;\, T_H(\mathrm{Re}\, t) = -\, d(s, \mathrm{Re}\, t)\}.$$

There are several elementary facts about Σ that will be needed later on when estimating the various h-microlocal pieces of the $N^{\mathbb{C}}(h)$-operator in the course of proving Proposition 8.1. The first observation is that in view of Lemma 7.3,

$$\Sigma = \{(r(s), q(y(s))) \in \partial\Omega \times H\} \tag{70}$$

and (70) is a C^ω-graph over $\partial\Omega$ in the product manifold. Moreover, one also has the following useful fact:

LEMMA **7.5.** *Assume that* $H \subset \Omega$ *is an interior curve and that* $\partial\Omega$ *is smooth and convex. Then,*

$$\Sigma \subset \{(r(s), q(\mathrm{Re}\, t)) \in \partial\Omega \times H;\, \partial_{\mathrm{Re}\, t}\partial_s \rho(s, \mathrm{Re}\, t) = 0\}.$$

Proof. This follows from the formula

$$\partial_s \partial_{\mathrm{Re}\, t} \rho(s, \mathrm{Re}\, t) = \langle \partial_s d(s, \mathrm{Re}\, t), T_H(\mathrm{Re}\, t) \rangle.$$

□

We denote the canonical transformation induced by the diffeomorphism $s \mapsto y(s)$ by

$$
\begin{aligned}
&\zeta_H : T^*\partial\Omega \to T^*H, \\
&\zeta_H(s, \sigma) = (y, \eta);\;\; y = y(s), \eta = d_s y(s)^{-1}\sigma.
\end{aligned}
\tag{71}
$$

7.3. **Taylor expansion of $\rho^{\mathbb{C}}(t,s)$ around glancing points.** Before analyzing the composite operator $N^{\mathbb{C}*}(h) a N^{\mathbb{C}}(h)$ we collect here some alternative asymptotic formulas for the real and imaginary parts of $\rho^{\mathbb{C}}(t,s)$ which are useful when $|t - y(s)| \ll 1$.

LEMMA **7.6.** *Let* $(t, y(s)) \in ([-\pi, \pi] + i[\frac{\delta}{2}, \delta]) \times [-\pi, \pi]$ *where* $y : [-\pi, \pi] \to [-\pi, \pi]$ *is the diffeomorphism in Lemma 7.3. Then for* $|t - y(s)| \leq \delta$ *and* $\delta > 0$ *sufficiently small,*

$$\operatorname{Re} \rho^{\mathbb{C}}(t, s) = |q(y(s)) - r(s)| - (\operatorname{Re} t - y(s)) \left(1 + \frac{1}{2}\kappa_H^2(y(s))|\operatorname{Im} t|^2\right) + \max_{\alpha+\beta=4} \mathcal{O}(|\operatorname{Re} t - y(s)|^\alpha |\operatorname{Im} t|^\beta),$$

$$\operatorname{Im} \rho^{\mathbb{C}}(t, s) = -\operatorname{Im} t + \frac{\kappa_H(y(s))}{2}(\operatorname{Re} t - y(s))^2 \operatorname{Im} t - \frac{1}{6}\kappa_H^2(y(s))(\operatorname{Im} t)^3 + \max_{\gamma+\delta=5} \mathcal{O}(|\operatorname{Re} t - y(s)|^\gamma |\operatorname{Im} t|^\delta).$$

Proof. The lemma follows from the formula

$$\rho^{\mathbb{C}}(t, s) = |q(y(s)) - r(s)| - (t - y(s)) + \frac{1}{6}\kappa_H^2(y(s))(t - y(s))^3 + \mathcal{O}(|t - y(s)|^4)$$

This inturn is a consequence of the Taylor expansion for $q^{\mathbb{C}}(t) - r(s) = q^{\mathbb{C}}(t) - q(y(s)) + q(y(s)) - r(s)$ around $t = y(s)$ in (62) using in addition the identities $\langle d(s, y(s)), T_H(y(s))\rangle = -1$ and $\langle d(s, y(s)), \nu_H(y(s))\rangle = 0$. □

7.4. **Weight function.** We compute in this section the asymptotic formula for the weight function, $S(t)$.

LEMMA **7.7.** *Let* $q^{\mathbb{C}}(t) \in H^{\mathbb{C}}(\delta) - H^{\mathbb{C}}(\delta/2)$ *with* $\delta > 0$ *sufficiently small. Then, the weight function*

$$S(t) = \operatorname{Im} t + \frac{1}{6}\kappa_H^2(\operatorname{Re} t)(\operatorname{Im} t)^3 + \mathcal{O}(|\operatorname{Im} t|^5),$$

Proof. We first consider the approximate critical point equation

$$\partial_s \langle T_H(t), d(s, t)\rangle = |\operatorname{Im} t|^4. \tag{72}$$

When $\operatorname{Im} t = 0$, (72) has the solution $s(\operatorname{Re} t) := y^{-1}(\operatorname{Re} t)$ in the notation of Lemma 7.3. Under the assumption that $\kappa_H > 0$, $\partial_s^2 \langle T_H(\operatorname{Re} t), d(s, \operatorname{Re} t)\rangle|_{s=s(\operatorname{Re} t)} \geq \frac{1}{C} > 0$ and so, by the analytic implicit function theorem, for $\delta > 0$ small, $s(\operatorname{Re} t)$ locally extends to a unique analytic $s = s(t)$, $t \in [-\pi, \pi] + i[\frac{\delta}{2}, \delta]$ solving (72). Substitute the identity $\langle d(s(\operatorname{Re} t), \operatorname{Re} t), T_H(\operatorname{Re} t)\rangle = -1 + |\operatorname{Im} t|^4$ into the formula (63) and also use that $\langle \nu_H(\operatorname{Re} t), d(s(\operatorname{Re} t), t)\rangle = |\operatorname{Im} t|^2$ and $\partial_s \langle d, \nu_H\rangle = -\langle d, T_H\rangle(1 - \langle T_H, d\rangle^2)^{-1/2}\, \partial_s \langle T_H, d\rangle$ both of which follow from the fact that $\langle \nu_H, d\rangle = \sqrt{1 - \langle T_H, d\rangle^2)}$. Since the last two terms in the $(\operatorname{Im} t)^3$-coefficient on the right hand side of (63) cancel, one gets that

$$\partial_s \operatorname{Im} \rho^{\mathbb{C}}(t, s)|_{s=s(t)} = \partial_s \langle T_H(\operatorname{Re} t), d(s, t)\rangle|_{s=s(t)} \left(\operatorname{Im} t + \mathcal{O}(|\operatorname{Im} t|^3)\right) + \mathcal{O}(|\operatorname{Im} t|^5) = \mathcal{O}(|\operatorname{Im} t|^5). \tag{73}$$

Finally, we compare (73) with exact critical point equation

$$\partial_s \operatorname{Im} \rho^{\mathbb{C}}(s, t) = 0. \tag{74}$$

Let $s = s^*(t)$ be the locally unique analytic solution to (74) with $\delta/2 < \operatorname{Im} t < \delta$ and $\delta > 0$ small. Then, again by the Taylor expansion in (63) and the implicit function theorem, it follows that

$$s(t) - s^*(t) = \mathcal{O}(|\operatorname{Im} t|^4). \tag{75}$$

Upon substituting these bounds back in (63) it follows that all terms except the one involving $\frac{1}{6}\kappa_H^2(\operatorname{Re} t)$ are absorbed into the $\mathcal{O}(|\operatorname{Im} t|^5)$-error and, in particular,

$$\operatorname{Im} \rho^{\mathbb{C}}(t, s^*(t)) = -\operatorname{Im} t - \frac{1}{6}\kappa_H^2(\operatorname{Re} t)(\operatorname{Im} t)^3 + \mathcal{O}(|\operatorname{Im} t|^5) \tag{76}$$

and this gives the asymptotic formula for $S(t) = \max_{s\in[-\pi,\pi]}(-\mathrm{Im}\,\rho^{\mathbb{C}}(t,s))$ to $\mathcal{O}(|\mathrm{Im}\,t|^5)$ error. □

Remark: One can repeat the same kind of argument to determine the expansion of $S(t)$ in $\mathrm{Im}\,t$ to arbitrary accuracy, but the terms rapidly become more cumbersome to compute.

The value of the weight function (ie. the maximizer of $-\mathrm{Im}\,\rho^{\mathbb{C}}$)is approximately attained when $s = y^{-1}(t) \in [-\pi,\pi]$ (see (75)). This suggests that the main piece of the the $N^{\mathbb{C}}(h)$ -operator (resp. $N^{\mathbb{C}*}(h)aN^{\mathbb{C}}(h)$ for any $a \in C_0^\infty(S_{2\varepsilon_\circ,2\pi})$ should come from $\Sigma^{\mathbb{C}}_{\varepsilon_\circ}$ (resp. $\Sigma^{\mathbb{C}}_{\varepsilon_\circ} \times \Sigma^{\mathbb{C}}_{\varepsilon_\circ}$), where

$$\Sigma^{\mathbb{C}}_{\varepsilon_\circ} := \{(t,s) \in S_{2\varepsilon_\circ,2\pi} \times [-\pi,\pi];\, |t-y(s)| < \varepsilon_\circ\}. \tag{77}$$

We will call $\Sigma^{\mathbb{C}}_{\varepsilon_\circ}$ the $\varepsilon_\circ$-*complex glancing set relative to* H in the parameter space $\{(t,s) \in S_{2\varepsilon_\circ,2\pi} \times [-\pi,\pi]\}$.

One of the first steps in the next section will be to show that the contribution to $N^{\mathbb{C}*}(h)aN^{\mathbb{C}}(h)$ coming from the complement, $\{t \in S_{2\varepsilon_\circ,2\pi}; |y(s)-t| \geq \varepsilon_\circ\}$ is of lower order in h in L^2-norm then the contribution coming from the complex glancing set $\Sigma^{\mathbb{C}}_{\varepsilon}$. This fact relies on some estimates for $\mathrm{Im}\,\rho^{\mathbb{C}}$ which we collect here. Assume that $|\mathrm{Im}\,t| \lessapprox \varepsilon_\circ$ and that $|\mathrm{Re}\,t - y(s)| \geq \varepsilon_\circ$. Then from (61), it follows that with $y = y(s)$,

$$\begin{aligned}
&\langle q^{\mathbb{C}}(\mathrm{Re}\,t + i\mathrm{Im}\,t) - r(y), q^{\mathbb{C}}(\mathrm{Re}\,t + i\mathrm{Im}\,t) - r(y)\rangle \\
&= |q(\mathrm{Re}\,t) - q(y)|^2 + 2i\mathrm{Im}\,t\langle q(\mathrm{Re}\,t) - r(y), T_H(\mathrm{Re}\,t)\rangle + \mathcal{O}(|\mathrm{Im}\,t|^2) \\
&\qquad + \langle q(\mathrm{Re}\,t) - q(y), q(y) - r(y)\rangle + |q(y) - r(y)|^2.
\end{aligned} \tag{78}$$

and so, taking square roots in (78) gives

$$\begin{aligned}
\big|\mathrm{Im}\,\rho^{\mathbb{C}}(\mathrm{Re}\,t + i\mathrm{Im}\,t, y)\big| &= \frac{|\mathrm{Im}\,t\langle q(\mathrm{Re}\,t) - r(y), T_H(\mathrm{Re}\,t)\rangle + \mathcal{O}(|\mathrm{Im}\,t|^2)|}{|q(\mathrm{Re}\,t) - r(y)|} \\
&= |\mathrm{Im}\,t||\langle d(y,\mathrm{Re}\,t), T_H(\mathrm{Re}\,t)\rangle| + \mathcal{O}(\mathrm{Im}\,t^2) \leq \frac{1}{C(\varepsilon_\circ)}|\mathrm{Im}\,t| + \mathcal{O}(|\mathrm{Im}\,t|^2),
\end{aligned} \tag{79}$$

with $C(\varepsilon_\circ) > 1$. The last estimate in (79) follows since $|\mathrm{Re}\,t - y(s)| \gtrapprox \varepsilon_\circ$ implies that $|\langle d(y(s),\mathrm{Re}\,t), T_H(\mathrm{Re}\,t)\rangle| \leq \frac{1}{C(\varepsilon_\circ)}$ with $C(\varepsilon_\circ) > 1$.

8. Proof of Proposition 6.1

In this section, we prove a somewhat more detailed version of Proposition 6.1; this amounts to a careful analysis of the conjugate operator $N^*(h)aN(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$. This entails several complications; most important of which is that it is only an h-pseudodifferential operator when h-microlocalized away from the glancing set (in the boundary case these are the tangential directions to the boundary). In quantum ergodicity of quantum ergodic restriction, these sets do not affect the limiting asymptotics and are therefore ignored [TZ2].

However, here the situation is very different. We are actually interested in the complexified operator $[e^{-S/h}N^{\mathbb{C}}(h)]^* a [e^{-S/h}N^{\mathbb{C}}(h)] : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ where $a \in C_0^\infty(H^{\mathbb{C}}_{\varepsilon_\circ})$ is supported in the annulus $H^{\mathbb{C}}_{2\varepsilon_\circ/3} - H^{\mathbb{C}}_{\varepsilon_\circ/6}$ (see subsection 1.2). In this case, it is precisely the glancing set Σ that determines the leading operator asymptotics. To analyze this operator we will need to make a further h-microlocal decomposition by splitting the complex near-glancing directions into the "near-real" and complimentary directions. Fortunately, the fact

that we are dealing with *complex* near-glancing sets (rather than real ones) actually simplifies the analysis of the microlocal complex $\varepsilon_\circ$ near-glancing piece of the $N^{\mathbb{C}^*}(h)aN^{\mathbb{C}}(h)$-operator, as long as the support of $a \in C_0^\infty(H^{\mathbb{C}}_{\varepsilon_\circ})$ lies outside an arbitrarily small neighbourhood of the real curve, H. The proof of the following Proposition essentially consists of carrying out the details of this h-microlocalization.

Proposition **8.1.** *Let $H \subset \Omega$ be a closed, strictly convex, interior real analytic curve. Let $N^{\mathbb{C}}(q^{\mathbb{C}}, r; h)$ be the holomorphic extension of $N(q, r; h)$ in the q variables to $H^{\mathbb{C}}_{\varepsilon_\circ}$ with the corresponding operator*

$$N^{\mathbb{C}}(h) : L^2(\partial\Omega; ds) \to L^2(H^{\mathbb{C}}_{\varepsilon_\circ};\, e^{-\frac{S(t)}{h}}\, dtd\bar{t}),$$

where $H^{\mathbb{C}}_{\varepsilon_\circ} = \{q^{\mathbb{C}}(t); |\operatorname{Im} t| \leq \varepsilon_\circ\}$. Let $a \in C_0^\infty(H^{\mathbb{C}}_{\varepsilon_\circ})$ with

$$supp\, a \subset \{q^{\mathbb{C}}(t) \in H^{\mathbb{C}}_{\varepsilon_\circ};\, \frac{\varepsilon_\circ}{6} \leq \operatorname{Im} t \leq \frac{5\varepsilon_\circ}{6}\}. \tag{80}$$

Then for $h \in (0, h_0(\varepsilon_\circ)]$, and $\varepsilon_\circ > 0$ sufficiently small, there exists an associated symbol $a_{\mathcal{G}} \in C_0^\infty(B^*\partial\Omega) \subset S^{0,-\infty}(T^*\partial\Omega \times (0, h_0])$ *such that*

$$h^{-1/2}\, N^{\mathbb{C}}(h)^* e^{-2S/h} a N^{\mathbb{C}}(h) = Op_h(a_{\mathcal{G}}) + R(h).$$

For $(s, \sigma) \in B^*\partial\Omega$, the symbol

$$a_{\mathcal{G}}(s, \sigma) = \frac{1}{\sqrt{2}} a(\operatorname{Re} t(s, \sigma), \operatorname{Im} t(s, \sigma))\, \kappa_H^{-2}(y(s))\, |\operatorname{Im} t(s, \sigma)|^{-1}\, \gamma^2(s, \sigma),$$

where, $\gamma(s, \sigma) = \sqrt{1 - |\sigma|^2}$ and

$$y(s) = \operatorname{Re} t(s, \sigma)(1 + \mathcal{O}(|\operatorname{Im} t(s, \sigma)|)),$$
$$\sigma = -\langle d(s, y(s)), T_{\partial\Omega}(s)\rangle + \frac{\kappa_H^2(y(s))}{2} d_s y(s) |\operatorname{Im} t(s, \sigma)|^2 (1 + \mathcal{O}(|\operatorname{Im} t(s, \sigma)|)). \tag{81}$$

Moreover, the remainder satisfies

$$\|R(h)\|_{L^2(\partial\Omega) \to L^2(\partial\Omega)} = \mathcal{O}(h^{1/2}).$$

Remark: Since $|\langle d(s, y(s)), T_{\partial\Omega}(s)\rangle| < 1$, it follows from (81) and the support assumptions on $a(\operatorname{Re} t, \operatorname{Im} t)$ in (80) that for $\varepsilon_\circ > 0$ small, $a_{\mathcal{G}} \in C_0^\infty(B^*\partial\Omega)$ (ie. has support disjoint from the tangential set $S^*\partial\Omega$).

Proof. We first cutoff near the glancing point $t = y(s)$ by introducing a cutoff function $\chi \in C_0^\infty(\mathbb{C})$ with $\chi(z) = 1$ when $|z| \leq \frac{\varepsilon_\circ}{2}$ and $\chi(z) = 0$ for $|z| \geq \varepsilon_\circ$. Here, $\varepsilon_\circ > 0$ is fixed but chosen arbitrarily small. We decompose the operator $N^{\mathbb{C}}(h)$ in various stages. First, we write

$$N^{\mathbb{C}}(h) = N_1^{\mathbb{C}}(h) + N_2^{\mathbb{C}}(h) + \mathcal{E}(h)$$

where,

$$N_1^{\mathbb{C}}(t, s; h) = Ch^{-1} e^{i\rho^{\mathbb{C}}(t,s)/h}\, \chi(|t - y(s)|)\, b(h^{-1}\rho^{\mathbb{C}}(t, s)), \tag{82}$$

and

$$N_2^{\mathbb{C}}(t, s; h) = Ch^{-1} e^{i\rho^{\mathbb{C}}(t,s)/h}\, (1 - \chi)(|t - y(s)|)\, b(h^{-1}\rho^{\mathbb{C}}(t, s)), \tag{83}$$

where $b(t)$ has an asymptotic expansion in inverse powers of t as $t \to \infty$, with leading term $\sim t^{-1/2}$ and recall the the diffeomorphism $y : [-\pi, \pi] \to [-\pi, \pi]$ is characterized by

the identity $\langle T_H(y(s)), d(s, y(s))\rangle = -1$. The operator $\mathcal{E}(h) : L^2(\partial\Omega) \to L^2(\partial\Omega)$ satisfies $\|\mathcal{E}(h)\|_{L^2\to L^2} = \mathcal{O}(h^\infty)$ and so is negligible.

9. Analysis of $N^{\mathbb{C}}(h)^* e^{-2S/h} a N^{\mathbb{C}}(h)$: Proof of Proposition 8.1

From (82) and (83) we make the decomposition

$$N^{\mathbb{C}}(h)^* a N^{\mathbb{C}}(h) = N_2^{\mathbb{C}}(h)^* a N_2^{\mathbb{C}}(h) + N_1^{\mathbb{C}}(h)^* a N_1^{\mathbb{C}}(h) + N_2^{\mathbb{C}}(h)^* a N_1^{\mathbb{C}}(h) + N_1^{\mathbb{C}}(h)^* a N_2^{\mathbb{C}}(h).$$

We first analyze the diagonal terms $N_1^{\mathbb{C}}(h)^* a N_1^{\mathbb{C}}(h)$ and $N_2^{\mathbb{C}}(h)^* a N_2^{\mathbb{C}}(h)$ and use Cauchy-Schwarz to estimate the off-diagonal term $N_k^{\mathbb{C}}(h)^* a N_l^{\mathbb{C}}(h)$ with $k \neq l$ at the end.

9.1. **Analysis of the $N_2^{\mathbb{C}*}(h) a N_2^{\mathbb{C}}(h)$-term: Reduction to the real case.** This piece of $N^{\mathbb{C}}(h)^* e^{-2S/h} a N^{\mathbb{C}}(h)$ is controlled by applying the a priori bound for the phase function in (79) and integrating-out the imaginary $\operatorname{Im} t$-variable. For future reference, we prove a slightly more general result for $N_2^{\mathbb{C}*}(h) a N_2^{\mathbb{C}}(h)$ than is needed in this paper. In particular, to control this part of the operator, we will simply assume that $a \in C_0^\infty(H_{\varepsilon_\circ}^{\mathbb{C}})$ (no annular support away from $\operatorname{Im} t = 0$ is needed). More precisely, we prove

Lemma **9.1.** *Let $H_{\varepsilon_\circ}^{\mathbb{C}} \supset H$ be a sufficiently small complex tube containing H. Then, for $h \in (0, h_0(\varepsilon_\circ)]$ sufficiently small and any $a \in C_0^\infty(H_{\varepsilon_\circ}^{\mathbb{C}})$,*

$$\|N_2^{\mathbb{C}}(h)^* e^{-2S/h} a N_2^{\mathbb{C}}(h)\|_{L^2\to L^2} = \mathcal{O}(h).$$

In the above and throughout this section, $L^2 := L^2(\partial\Omega; dr)$.

Proof.

$$\begin{aligned} N_2^{\mathbb{C}*} e^{-2S/h} a N_2^{\mathbb{C}}(s, s'; h) = \int\int_{S_{\varepsilon_\circ,\pi}} & (1-\chi)(|t - y'(s)|)\ (1-\chi)(|t - y(s)|) \\ & \times a(q^{\mathbb{C}}(t), \overline{q^{\mathbb{C}}(t)})\ \overline{N^{\mathbb{C}}(t, s'; h)}\ N^{\mathbb{C}}(t, s; h)\, e^{-2S(t)h}\ dt d\bar{t}, \end{aligned} \tag{84}$$

we reduce the analysis of the kernel in (84) to the real case and then apply a Kuranishi change of variables argument to show that it is an h-pseudodifferential operator of order -1. We will make this reduction by carrying out the $\operatorname{Im} t$-integration first in (84) and then estimate the result by making a Taylor expansion and an integration by parts.

Since we carry out the $\operatorname{Im} t$-integral in (84) first, it is useful to identify the iterated integral

$$\begin{aligned} I_a(\operatorname{Re} t, y, y'; h) := (1-\chi)(|t-y|)\ (1-\chi)(|t-y'|) \int_{|\operatorname{Im} t| \leq \varepsilon_\circ} & \overline{N^{\mathbb{C}}(t, y'; h)} N^{\mathbb{C}}(t, y; h) \\ & \times a(q^{\mathbb{C}}(t), \overline{q^{\mathbb{C}}(t)})\, e^{-2S(t)/h}\, d\operatorname{Im} t. \end{aligned} \tag{85}$$

From the Hankel function formula for $N^{\mathbb{C}}(h)$ in (18), for all $(t, s) \in S_{2\varepsilon_\circ, 2\pi} \times [-\pi, \pi]$ with $\varepsilon_\circ > 0$ sufficiently small, $|\rho^{\mathbb{C}}(t, s)| \geq \frac{1}{C(\varepsilon_\circ)} > 0$. Consequently, the $N_2^{\mathbb{C}}(h)$ kernel has a WKB-type polyhomogeneous expansion of the form

$$\begin{aligned} N_2^{\mathbb{C}}(t, s; h) = b_0 h^{-1} e^{i\rho^{\mathbb{C}}(t,s)/h} \left(\frac{h}{\rho^{\mathbb{C}}(t, s)}\right)^{1/2} \end{aligned} \tag{86}$$

$$\times A(t, s) \left(1 + b_1(t, s)\frac{h}{\rho^{\mathbb{C}}(t, s)} + b_2(t, s)\left(\frac{h}{\rho^{\mathbb{C}}(t, s)}\right)^2 + ...\right) + \mathcal{O}(e^{2S(t)/h} h^\infty).$$

where, $A(t,s) = \langle \nu_{\partial\Omega}(s), \rho^{\mathbb{C}}(t,s)\rangle$ and $b_j(t,s); j=1,2,...$ are holomorphic in the t-variable provided $t \in S_{2\varepsilon_\circ,2\pi}$.

By Taylor expansion of the amplitude in (86) centered at $\operatorname{Im} t = 0$,

$$\left(\frac{h}{\rho^{\mathbb{C}}(t,s)}\right)^{1/2} = \left(\frac{h}{|q(\operatorname{Re} t) - r(s)|}\right)^{1/2} \left[1 + \operatorname{Im} t \, \frac{\partial_{\operatorname{Im} t}\rho^{\mathbb{C}}(t,s)|_{\operatorname{Im} t=0}}{|q(\operatorname{Re} t) - r(s)|} + \mathcal{O}(|\operatorname{Im} t|^2)\right]. \quad (87)$$

Substitution of the expansions (86), (87) and corresponding ones for $\overline{N^{\mathbb{C}}(t,s';h)}$ into (85) shows that $I_a(\operatorname{Re} t, s, s'; h)$ has an expansion of the form:

$$C\,h^{-2}\; e^{i[|q(\operatorname{Re} t)-r(s)|-|q(\operatorname{Re} t)-r(s')|]/h} \left(\frac{|q(\operatorname{Re} t)-r(s)|}{h}\right)^{-1/2} \left(\frac{|q(\operatorname{Re} t)-r(s')|}{h}\right)^{-1/2} (1-\chi)(|t-y'(s)|) \quad (88)$$

$$\times (1-\chi)(|t-y(s)|) a(q(\operatorname{Re} t), \overline{q(\operatorname{Re} t)}) \int_{|\operatorname{Im} t| \leq \varepsilon_\circ} e^{i\Psi(t,s,s')/h} \left(1 + c_1(\operatorname{Re} t, s, s'; h)\operatorname{Im} t\,\right) d\operatorname{Im} t$$

$$+ \int_{|\operatorname{Im} t|\leq\varepsilon_\circ} e^{i\Psi(t,s,s')/h} \mathcal{O}(|\operatorname{Im} t|^2))\; d\operatorname{Im} t,$$

where, the effective phase

$$i\Psi(t,s,s') := i\rho^{\mathbb{C}}(t,s) - i\rho^{\mathbb{C}}(\bar{t},s') - i\rho(\operatorname{Re} t, s) + i\rho(\operatorname{Re} t, s') - 2S(t). \quad (89)$$

In (88), $c_1 \in C^{\omega}(\mathbb{R}^3/(2\pi\mathbb{Z})^3)$ with $c_1 \sim \sum_{k=0}^{\infty} c_{1k}h^k$ and the $\mathcal{O}(|\operatorname{Im} t|^2)$-error is uniform in $h \in (0, h_0]$.

We split the $d\operatorname{Im} t$-integration in (88) up into the pieces $\int_0^\infty ...$ (resp. $\int_{-\infty}^0 ...$) and treat each case separately. Since the analysis is the same in either case, we continue to assume that $\operatorname{Im} t \geq 0$ here and will estimate the $\int_0^\infty ... d\operatorname{Im} t$-piece.

From the convexity of the boundary and the estimate (79), for $0 \leq \operatorname{Im} t \leq 2\varepsilon_\circ$ with $\varepsilon_\circ > 0$ sufficiently small and $\min(|t-y|, |t-y'|) \geq \frac{\varepsilon_\circ}{2}$, there exists $C(\varepsilon_\circ) > 1$ such that

$$|\operatorname{Re}\,\left(\, i\rho^{\mathbb{C}}(t,y) - i\rho^{\mathbb{C}}(\bar{t},y')\,\right)| \leq \frac{\operatorname{Im} t}{C(\varepsilon_\circ)}.$$

From Lemma 7.7, $S(t) = \operatorname{Im} t + \mathcal{O}(|\operatorname{Im} t|^3)$ and so for $(t,s,s') \in S_{\varepsilon_\circ,\pi} \times \operatorname{Supp}(1-\chi)(|y(\cdot) - t|) \times \operatorname{Supp}(1-\chi)(|y(\cdot)-t|)$,

$$|\partial_{\operatorname{Im} t}\Psi(t,y,y')| \geq (1 - C'(\varepsilon_\circ)^{-1}) > 0, \quad (90)$$

where, the right hand side in (90) is uniform in all variables and $1 < C'(\varepsilon_\circ)$ for $\varepsilon_\circ > 0$ sufficiently small.

Next, we rescale the $\operatorname{Im} t$-variable and put $\tau = h^{-1}\operatorname{Im} t$. Then, from (90) and the expansion in (88) we get that $\int_0^{\varepsilon_\circ} e^{i\Psi(t,s,s')/h}\left(1 + c_1(\operatorname{Re} t, s, s'; h)\operatorname{Im} t\right) d\operatorname{Im} t$ equals

$$h\int_0^{\varepsilon_\circ/h} e^{ib_1(\operatorname{Re} t,s,s')\tau}\; e^{\sum_{j\geq 2} b_j(\operatorname{Re} t,s,s')\tau^j h^{j-1}} \left(1 + c_1(\operatorname{Re} t,s,s')\tau h + c_2(\operatorname{Re} t,s,s')\tau^2h^2 + ...\right) d\tau. \quad (91)$$

From (90) it follows that $\operatorname{Im} b_1(\operatorname{Re} t, s, s') \geq 1 - C'(\varepsilon_\circ)^{-1} > 0$ in (91) and so by Taylor expansion of $\exp(\sum_{j\geq 2} b_j(\operatorname{Re} t, s, s')\tau^j h^{j-1})$ in (91) using the bound $|e^x - \sum_{j=0}^N \frac{x^j}{j!}| \leq \frac{|x|^N}{N!}$, it

follows that

$$\int_0^{\varepsilon_\circ} e^{i\Psi(\mathrm{Re}\,t,\mathrm{Im}\,t,s,s')/h}\,d\mathrm{Im}\,t = h\int_0^{\varepsilon_\circ/h} e^{ib_1(\mathrm{Re}\,t,s,s')\tau}\,\left(\sum_{l=0}^{N}\sum_{k\le l} c_{kl}(\mathrm{Re}\,t,s,s')h^k\tau^l\right)\,d\tau + \mathcal{O}(h^{N+1}),$$

where $c_{kl}\in C^\omega(\mathbb{R}^3/(2\pi\mathbb{Z})^3)$ and $c_{00}=1$.

Since $\mathrm{Im}\,b_1>0$ one computes for $k\le l$

$$\int_0^{\delta(\varepsilon_\circ)/h} e^{ib_1(\tau,s,s')\tau}h^k\tau^l d\tau = \int_0^\infty e^{ib_1(\mathrm{Re}\,t,s,s')\tau}h^k\tau^l d\tau + \mathcal{O}(e^{-C_7/h}) = \tilde{b}_{kl}(\mathrm{Re}\,t,s,s')h^{k+l}+\mathcal{O}(e^{-C_7/h}), \tag{92}$$

where $\tilde{b}_{kl}\in C^\omega$ and $C_7>0$. From (92) it follows that

$$\begin{aligned} N_2^{\mathbb{C}*}(h)e^{-2S/h}aN_2^{\mathbb{C}}(h)(s,s';h) &= \int_{-\infty}^\infty e^{i\psi(\mathrm{Re}\,t,s,s')/h}\,a(q(\mathrm{Re}\,t),\overline{q(\mathrm{Re}\,t)})\,(1+\sum_{j=1}^N d_j(\mathrm{Re}\,t,s,s')h^j) \\ &\times(1-\chi)(|\mathrm{Re}\,t-y(s)|)\,(1-\chi)(|\mathrm{Re}\,t-y'(s)|)\;d\mathrm{Re}\,t+\mathcal{O}_N(h^{N+1}), \end{aligned} \tag{93}$$

where,

$$\psi(\mathrm{Re}\,t,s,s') := |q(\mathrm{Re}\,t)-r(s)|-|q(\mathrm{Re}\,t)-r(s')|. \tag{94}$$

As a consequence of (93), except for some innocuous amplitude terms, we have reduced the analysis of $N_2^{\mathbb{C}*}(h)e^{-2S/h}aN_2^{\mathbb{C}}(h)(y,y';h)$ to the study of compositions of *real* single layer potentials with their adjoints (see [HZ]).

We claim that (93) is the kernel of an h-pseudodifferential operator of order -1. Indeed, since $\min(|y(s)-\mathrm{Re}\,t|,|y'(s)-\mathrm{Re}\,t|)\ge\frac{\varepsilon_\circ}{2}$ it is clear that $\psi\in C^\infty$ and so, since $\psi(\mathrm{Re}\,t,s,s)=0$, by a first-order Taylor expansion,

$$\psi(\mathrm{Re}\,t,s,s')=(s-s')\cdot\eta(\mathrm{Re}\,t,s,s'), \tag{95}$$

where, by Lemma 7.5 we know that for $|y(s)-\mathrm{Re}\,t|\ge\frac{\varepsilon_\circ}{2}$,

$$\partial_{\mathrm{Re}\,t}\partial_s(|q(\mathrm{Re}\,t)-r(s)|)>C_1>0$$

and so it follows by Taylor's theorem that for $|s-s'|\le\varepsilon_\circ$ with $\varepsilon_\circ>0$ small enough,

$$|\partial_{\mathrm{Re}\,t}\,\eta(\mathrm{Re}\,t,s,s')|\ge C_2>0. \tag{96}$$

Note that when $|s-s'|\ge\varepsilon_\circ$, by convexity and the fact that $\min(|y(s)-\mathrm{Re}\,t|,|y'(s)-\mathrm{Re}\,t|)\ge\frac{\varepsilon_\circ}{2}>0$,

$$|\,\partial_{\mathrm{Re}\,t}\,(|q(\mathrm{Re}\,t)-r(s)|-|q(\mathrm{Re}\,t)-r(s')|)\,| = |d(s,\mathrm{Re}\,t)-d(s',\mathrm{Re}\,t)|\ge C_{10}>0. \tag{97}$$

So, in the latter case, an integration by parts in $\mathrm{Re}\,t$ in (93) shows that modulo $\mathcal{O}(h^\infty)$-errors, one can insert the additional cutoff function $\chi_{\varepsilon_\circ}(|s-s'|)=\chi(\varepsilon_\circ^{-1}|s-s'|)$ where $\varepsilon_\circ>0$ is arbitrarily small. But then, one can apply the Jacobian estimate in (96) to make the change of variables $\mathrm{Re}\,t\mapsto\eta(\mathrm{Re}\,t;s,s')$ in (93) and get that

$$\begin{aligned} &N_2^{\mathbb{C}*}(h)e^{-2S/h}aN_2^{\mathbb{C}}(h)(s,s') \\ &= \int_{\mathbb{R}} e^{i(s-s')\eta/h}\,a(q(\mathrm{Re}\,t(s,s',\eta)),\overline{q(\mathrm{Re}\,t(s,s',\eta))})\,(d_0(s,s',\eta)+d_1(s,s',\eta)h+\ldots)\,d\eta, \end{aligned}$$

with $d_j\in C^\omega$. Thus, $N_2^{\mathbb{C}*}(h)aN_2^{\mathbb{C}}(h)\in Op_h(S^{-1,-\infty}(T^*\partial\Omega))$ and so by L^2-boundedness, we are done. ☐

Remark: In the special case of interest in Theorem 1.2, $a(t)$ is supported in an annular subset of $H^{\mathbb{C}}_{\varepsilon_\circ}$ with supp $a \subset H^{\mathbb{C}}_{2\varepsilon_\circ/3} - H^{\mathbb{C}}_{\varepsilon_\circ/6}$. In that case, it follows from the estimate in (78) that Lemma 9.1 can be improved and one gets that for appropriate $C(\varepsilon_\circ) > 0$ and $\varepsilon_\circ > 0$ small,

$$N_2^{\mathbb{C}}(h)^* e^{-2S/h} a N_2^{\mathbb{C}}(h)(s,s') = \mathcal{O}(e^{-C(\varepsilon_\circ)/h}) \tag{98}$$

uniformly for $(s,s') \in \partial\Omega \times \partial\Omega$. The same is true for partial derivatives.

9.2. **Estimate for $N_1^{\mathbb{C}*}(h)e^{-2S/h}aN_1^{\mathbb{C}}(h)$: The dominant term.** In contrast with the analysis for $N_2^{\mathbb{C}*}(h)e^{-2S/h}aN_2^{\mathbb{C}*}(h)$ above, in this case we carry out the $\mathrm{Re}\, t$-integration first. After decomposing the resulting integral into two further pieces (depending on whether $|\mathrm{Re}\, t - y(s)|$ or $|\mathrm{Im}\, t|$ dominates when $|t - y(s)| \leq \varepsilon_\circ$) and using the strict convexity of $H \subset \Omega$, we apply the method of steepest descent to expand the $\mathrm{Re}\, t$-integral. The remaining imaginary coordinate $\mathrm{Im}\, t$ then behaves roughly like a frequency variable in the oscillatory integral representation of an h-pseudodifferential operator of order $-\frac{1}{2}$ (here, we again use that H is strictly convex).

We decompose the $N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h)$ operator further as follows: Let $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ be a cutoff equal to 1 on a ball of radius 1/2 and zero outside the ball of radius 1. Also, to simplify the writing, we abuse notation and write $a(t)$ for $a(q^{\mathbb{C}}(t), \overline{q^{\mathbb{C}}(t)})$. We define the operators $N_1^{11}(h)$ and $N_1^{22}(h)$ with Schwartz kernels

$$\begin{aligned} &N_1^{11}(h)(y,y';a) \\ &= \int\int_{S_{\varepsilon_\circ,\pi}} e^{-2S(t)/h} N_1^{\mathbb{C}}(h)^*(y,t)\, a(t)\, \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y|}{|\mathrm{Im}\, t|}\right)\ \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y'|}{|\mathrm{Im}\, t|}\right) N_1^{\mathbb{C}}(h)(t,y')\, dt\, d\bar{t}, \end{aligned} \tag{99}$$

$$\begin{aligned} &N_1^{22}(h)(y,y;a) \\ &= \int\int_{S_{\varepsilon_\circ,\pi}} e^{-2S(t)/h} N_1^{\mathbb{C}}(h)^*(y,t)\, a(t)\, (1-\tilde{\chi})\left(\frac{|\mathrm{Re}\, t - y|}{|\mathrm{Im}\, t|}\right)\ (1-\tilde{\chi})\left(\frac{|\mathrm{Re}\, t - y'|}{|\mathrm{Im}\, t|}\right) N_1^{\mathbb{C}}(h)(t,y')\, dt\, d\bar{t}, \end{aligned} \tag{100}$$

and with mixed terms $N_1^{12}(h;a)$ and $N_1^{21}(h;a)$ defined in the obvious way so that

$$N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h) = N_1^{11}(h;a) + N_1^{22}(h;a) + N_1^{12}(h;a) + N_1^{21}(h;a).$$

Just as before, we control the mixed terms N_1^{12} and N_1^{21} using Cauchy Schwarz and the estimates for the diagonal terms, and it will suffice to analyze the diagonal terms N_1^{11} and N_1^{22}.

9.2.1. *Analysis of the $N_1^{11}(h)$-term: Reduction to normal form.* Our aim here is to reduce the operator $N_1^{11}(h)$ by suitable change of variables in $(\mathrm{Re}\, t, \mathrm{Im}\, t)$ to a normal form and then, by an application of analytic stationary phase ([Ho1] Theorem 7.7.12), we show that the normal form operator is in $Op_h(S^{-1/2,-\infty}(T^*\partial\Omega))$.

First, we recall the asymptotic formulas for $\mathrm{Re}\,(i\Psi)$ and $\mathrm{Im}\,(i\Psi)$ where Ψ is the effective phase function in (89). From Lemma 7.6,

$$\begin{aligned}\mathrm{Re}\,[i\Psi(t,s,s')] = &-\frac{\kappa_H^2(y(s))}{2}(\mathrm{Re}\,t - y(s))^2 \mathrm{Im}\,t + \max_{\gamma+\delta=4,\delta\geq 1} \mathcal{O}(|\mathrm{Re}\,t - y(s)|^{\gamma}|\mathrm{Im}\,t|^{\delta}) \\ &- \frac{\kappa_H^2(y(s'))}{2}(\mathrm{Re}\,t - y(s'))^2 \mathrm{Im}\,t + \max_{\gamma+\delta=4,\delta\geq 1} \mathcal{O}(|\mathrm{Re}\,t - y(s')|^{\gamma}|\mathrm{Im}\,t|^{\delta}).\end{aligned} \tag{101}$$

In (101) we note that (see Lemma 7.7) the terms $\frac{1}{6}\kappa_H^2(y(s))(\mathrm{Im}\,t)^3 + \frac{1}{6}\kappa_H^2(y(s'))(\mathrm{Im}\,t)^3$ get cancelled by the cubic term in $\mathrm{Im}\,t$ term appearing in the expansion of $S(t)$. Similarly,

$$\begin{aligned}&\mathrm{Im}\,[i\Psi(t,s,s')] \\ &= |q(y(s)) - r(s)| - (\mathrm{Re}\,t - y(s))\left(1 + \frac{1}{2}\kappa_H^2(y(s))|\mathrm{Im}\,t|^2\right) + \frac{\kappa_H^2(y(s))}{6}(\mathrm{Re}\,t - y(s))^3 \\ &+ \max_{\alpha+\beta=4} \mathcal{O}(|\mathrm{Re}\,t - y(s)|^{\alpha}|\mathrm{Im}\,t|^{\beta}) \\ &- |q(y(s')) - r(s')| + (\mathrm{Re}\,t - y(s'))\left(1 + \frac{1}{2}\kappa_H^2(y(s'))|\mathrm{Im}\,t|^2\right) - \frac{\kappa_H^2(y(s'))}{6}(\mathrm{Re}\,t - y(s'))^3 \\ &+ \max_{\alpha+\beta=4} \mathcal{O}(|\mathrm{Re}\,t - y(s')|^{\alpha}|\mathrm{Im}\,t|^{\beta}).\end{aligned} \tag{102}$$

Substitution of the identity $\langle T_H(y(s)), d(s, y(s))\rangle = -1$ and second-order Taylor expansion around $y = y'$ in (102) gives

$$\begin{aligned}\mathrm{Im}\,\left(i\Psi(\mathrm{Re}\,t; \mathrm{Im}\,t, s, s')\right) =& (y - y')\,\left(\langle T_H(y(s)), d(s, y(s))\rangle - (\partial_s y(s))^{-1}\langle T_{\partial\Omega}(s), d(s, y(s))\rangle\right) \\ &+ (y - y')\left(1 + \frac{\kappa_H^2(y(s))}{2}|\mathrm{Im}\,t|^2 + \mathcal{O}(|\mathrm{Im}\,t|^3)\right) + \mathcal{O}(|y - y'|^2). \\ =& (y - y')\,(-\partial_s y(s)^{-1}\langle T_{\partial\Omega}(s), d(s, y(s))\rangle \\ &+ \frac{\kappa_H^2(y(s))}{2}|\mathrm{Im}\,t|^2 + \mathcal{O}(|\mathrm{Im}\,t|^3)) + \mathcal{O}(|y - y'|^2) \\ = (s - s')&\left(-\langle T_{\partial\Omega}(s), d(s, y(s))\rangle + d_s y(s)\frac{\kappa_H^2(y(s))}{2}|\mathrm{Im}\,t|^2 + \mathcal{O}(|\mathrm{Im}\,t|^3)\right) + \mathcal{O}(|s - s'|^2).\end{aligned} \tag{103}$$

For the error term in (103), we have used the constraints $\max(|\mathrm{Re}\,t - y(s)|, |\mathrm{Re}\,t - y(s')|) = \mathcal{O}(|\mathrm{Im}\,t|)$ and also note that the $\mathcal{O}(|y - y'|^2)$-term appearing on the RHS of (103) is independent of the t-variables since it arises for the second-order Taylor expansion of the real-valued function $|q(y(s)) - r(s)|$ around $s = s'$.

9.3. **Normal form for the phase function** $i\Psi(t,s,s')$**.** We now reduce the computation of the principal term $N_1^{11}(h)$ to a specific normal form by applying a series of changes of variables in the $(\mathrm{Re}\,t, \mathrm{Im}\,t)$-coordinates. Given $(t, y(s)) \in S_{\varepsilon_\circ,\pi} \times [-\pi, \pi]$ we claim that near any point $t_0 \in S_{\varepsilon_\circ,\pi}$ with $\varepsilon_\circ > 0$ sufficiently small, one can find a locally a real-valued analytic function $f(\mathrm{Re}\,t, \mathrm{Im}\,t)$ satisfying

$$S(\mathrm{Re}\,t, \mathrm{Im}\,t) = \mathrm{Re}\,[\; i\rho^{\mathbb{C}}(t, f(\mathrm{Re}\,t, \mathrm{Im}\,t))\,]. \tag{104}$$

with

$$f(\mathrm{Re}\, t, \mathrm{Im}\, t) = \mathrm{Re}\, t + \mathcal{O}(|\mathrm{Im}\, t|). \tag{105}$$

To prove (104) and also (105), consider the real analytic function $g \in C^{\omega}(S_{\varepsilon_\circ,\pi} \times [-\pi,\pi])$ defined by

$$g(t,y) := \frac{\mathrm{Re}\ i\rho^{\mathbb{C}}(t,y)}{\mathrm{Im}\, t}. \tag{106}$$

We would like to solve

$$\partial_y\, g(t,y) = 0. \tag{107}$$

where, from the Taylor expansion in (101), we have the initial condition

$$\partial_y g(t,y)|_{\mathrm{Re}\, t = y, \mathrm{Im}\, t = 0} = 0.$$

Thus, (104) follows from the Implicit Function Theorem applied to (107), since from (101) and the strict convexity of H, (ie. $\kappa_H > 0$) we get that for $\varepsilon_\circ > 0$ sufficiently small,

$$\partial_y^2 g(t,y) \leq -\kappa_H^2(y) + \mathcal{O}(\varepsilon_\circ^2)$$

since $|\mathrm{Re}\, t - y(s)| \leq \varepsilon_\circ$ and $|\mathrm{Im}\, t| \leq \varepsilon_\circ$. So $y(s) = f(t)$ is a maximum for $\mathrm{Re}\, i\rho^{\mathbb{C}}(t,\cdot)$. By definition of $S(t)$, it is clear that

$$\mathrm{Re}\ [i\Psi](t,s,s')] \leq 0. \tag{108}$$

Given (104) we also have for all $t \in S_{\varepsilon_\circ,\pi}$,

$$\mathrm{Re}\ [i\Psi(t; f(t), f(t))] = 0. \tag{109}$$

To simplify notation in (109) and the following, we identify the complex variable t with the real 2-tuple $(\mathrm{Re}\, t, \mathrm{Im}\, t)$ in the argument of $i\Psi$. The pair of coordinates $(f(t), f(t))$ occupy the (y,y') coordinate slots. By definition $S(t) = \max_{y \in [0,2\pi]} \mathrm{Re}\, i\rho^{\mathbb{C}}(t,y)$ so that $\partial_y \mathrm{Re}\ [i\rho^{\mathbb{C}}(t,y)]|_{y=f(t)} = 0$. By differentiating (109) in $\mathrm{Re}\, t$ it follows that for any $t \in S_{\varepsilon_\circ,\pi}$,

$$\partial_{\mathrm{Re}\, t}\, \mathrm{Re}\, [i\Psi](t, f(t), f(t)) = 0. \tag{110}$$

Since $\mathrm{Im}\, i\Psi(t,s,s) = 0$, the identity $\partial_{\mathrm{Re}\, t}\, [\mathrm{Im}\, i\Psi](t, f(t), f(t)) = 0$ is automatic and so,

$$\partial_{\mathrm{Re}\, t}\, [i\Psi](t; f(t), f(t)) \;=\; 0. \tag{111}$$

Since $i\Psi(t,s,s') \in C^{\omega}(S_{\varepsilon_\circ,\pi} \times \mathbb{R}^2/(2\pi\mathbb{Z})^2)$ and $H \subset \Omega$ is strictly convex, from (101),

$$|\partial_{\mathrm{Re}\, t}^2 (i\Psi)| \geq \left[\kappa_H^2(y(s)) + \kappa_H^2(y'(s))\right]\ |\mathrm{Im}\, t| + \mathcal{O}(\varepsilon_\circ^2 |\mathrm{Im}\, t|) \geq C(\varepsilon_\circ) |\mathrm{Im}\, t|,$$

where $C(\varepsilon_\circ) > 0$ with $\varepsilon_\circ > 0$ sufficiently small. Differentiating (110) yet again in $\mathrm{Re}\, t$ gives

$$\partial_{\mathrm{Re}\, t}^2 \mathrm{Re}\, [i\Psi](t, f(t), f(t)) + 2\, \partial_y \partial_{\mathrm{Re}\, t} \mathrm{Re}\, [i\Psi](t, f(t), f(t)) \cdot \partial_{\mathrm{Re}\, t} f(t) = 0. \tag{112}$$

In view of the Taylor expansion (101), this simplifies to

$$2\kappa_H(\mathrm{Re}\, t)^2 |\mathrm{Im}\, t| + \mathcal{O}(|\mathrm{Re}\, t - f(t)||\mathrm{Im}\, t|) + \mathcal{O}(|\mathrm{Im}\, t|^2) - \; 2\kappa_H(\mathrm{Re}\, t)^2 |\mathrm{Im}\, t| \; \cdot \partial_{\mathrm{Re}\, t} f(t) = 0.$$

By dividing the last equation through by $\kappa_H(\mathrm{Re}\, t)^2 \mathrm{Im}\, t$, and solving for $\partial_{\mathrm{Re}\, t} f$, one gets that $f(t) = \mathrm{Re}\, t + \mathcal{O}(|\mathrm{Im}\, t|)$. By the same identity,

$$\partial_{\mathrm{Re}\, t} f(t) = 1 + \mathcal{O}(|\mathrm{Im}\, t|). \tag{113}$$

Given (113), we make the change of variables $(\mathrm{Re}\, t, \mathrm{Im}\, t) \mapsto (f(t), \mathrm{Im}\, t)$ in the tubular parameter domain $S_{\delta(\varepsilon_\circ),2\pi}$ with $\delta(\varepsilon_\circ) > 0$ sufficiently small. To reduce to normal form for $i\Psi$ we define the new variable

$$\tau_1(\mathrm{Re}\, t, \mathrm{Im}\, t) = f(t) = \mathrm{Re}\, t(1 + \mathcal{O}(\mathrm{Im}\, t)),$$
$$\partial_{\mathrm{Re}\, t}\tau_1 = 1 + \mathcal{O}(\mathrm{Im}\, t). \tag{114}$$

The complimentary variable τ_2 is defined by writing

$$\mathrm{Im}\,[i\Psi](t, s, s') = (y - y') \cdot \tau_2, \tag{115}$$

where from (103) we know that

$$\tau_2 = -\langle T_{\partial\Omega}(s), d(s, y(s))\rangle + d_s y(s) \frac{\kappa_H^2(y(s))}{2} |\mathrm{Im}\, t|^2 + \mathcal{O}(|\mathrm{Im}\, t|^3),$$
$$\partial_{\mathrm{Im}\, t}\tau_2 = d_s y(s)\, \kappa_H^2(y) \mathrm{Im}\, t + \mathcal{O}(|\mathrm{Im}\, t|^2), \tag{116}$$

since the last error term in (103) is independent of the t-variables. From (114) and (115) we derive the following normal form for the phase function $i\Psi$. By possibly shrinking $\varepsilon_\circ > 0$ and to simplify notation, we simply put $\delta(\varepsilon_\circ) = \varepsilon_\circ$ with the understanding that $\varepsilon_\circ > 0$ is to be taken sufficiently small.

LEMMA **9.2.** *In terms of the new coordinates (τ_1, τ_2) in $H^{\mathbb{C}}_{\varepsilon_\circ}$ defined in (114) and (115), it follows that the real part*

$$\mathrm{Re}\,[i\Psi](t(\tau_1, \tau_2); s, s') = -\alpha(\tau_1, \tau_2) \left[|y(s) - \tau_1|^2 + |y(s') - \tau_1|^2 + \mathcal{O}(|y(s) - \tau_1|^3) + |y(s') - \tau_1|^3) \right],$$

where, $\alpha(\tau_1, \tau_2) = [\, \kappa_H^2(t(\tau_1, \tau_2)) + \mathcal{O}(\varepsilon_\circ)\,]\mathrm{Im}\, t.$

The imaginary part

$$\mathrm{Im}\,[i\Psi](t(\tau_1, \tau_2); s, s') = (y - y')\tau_2,$$

where,

$$\tau_2(t(\tau_1, \tau_2); s, s') = -\partial_s y(s)^{-1} \langle T_{\partial\Omega}(s), d(y(s), s)\rangle + \frac{\kappa^2(y(s))}{2} |\mathrm{Im}\, t|^2 + \mathcal{O}(|\mathrm{Im}\, t|^3) + \mathcal{O}(|s - s'|).$$

Proof. The formula for $\mathrm{Re}\,(i\Psi)$ follows from the Taylor expansion in (101) plus the formula for the second derivative in (112). The formula for $\mathrm{Im}\,(i\Psi)$ inturn follows from (103). □

From Lemma 9.2 it follows that

$$\partial_{\mathrm{Im}\, t}\tau_2 = \kappa^2(y) \mathrm{Im}\, t + \mathcal{O}(|\mathrm{Im}\, t|^2). \tag{117}$$

So, since $a(t) \in C_0^\infty(\{t; \frac{\varepsilon_\circ}{6} < |\mathrm{Im}\, t| < \frac{2\varepsilon_\circ}{3}\})$ it follows from (117) and the derivative computation in (117) that for the change of variables $(\mathrm{Re}\, t, \mathrm{Im}\, t) \mapsto (\tau_1, \tau_2)$,

$$J(t; y, y') := \left| \frac{\partial(\tau_1, \tau_2)}{\partial(\mathrm{Re}\, t, \mathrm{Im}\, t)} \right| = \kappa_H^2(y) \cdot |\mathrm{Im}\, t| + \mathcal{O}(|\mathrm{Im}\, t|^2) \geq C(\delta(\varepsilon_\circ))|\mathrm{Im}\, t|. \tag{118}$$

Remark: It is at this point that we need the non-vanishing curvature condition $\kappa_H \neq 0$ to ensure that the Jacobian $J(t; y, y')$ is non-vanishing in (118).

We summarize our analysis so far in the following

LEMMA **9.3.** *Let $a \in C_0^\infty(H^{\mathbb{C}}_{2\varepsilon_\circ/3} - H^{\mathbb{C}}_{\varepsilon_\circ/6})$ Then for $\varepsilon_\circ > 0$ small enough and $h \in (0, h_0(\varepsilon_\circ)]$, the kernel $N_1^{11}(h)(y,y';a)$ equals*

$$(2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left[i\,(y-y')\tau_2 - \beta_1(y-\tau_1)^2 - \beta_2(y'-\tau_1)^2\right]/h \;\, a(t(\tau_1,\tau_2))\, \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y]|}{|\mathrm{Im}\, t|}\right) \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y']|}{|\mathrm{Im}\, t|}\right)$$

$$\times \rho(\tau_1,\tau_2;y,y';h')\, d\tau_1\, d\tau_2.$$

Here,

$$\begin{aligned}\rho(\tau_1,\tau_2;y,y';h) &= J^{-1}(t(\tau);y,y')\, b(t(\tau);y,y';h) \\ &= \frac{\kappa_H^{-2}(y(s))}{\sqrt{2}}\, |\mathrm{Im}\, t(\tau)|^{-1}\, b(t(\tau);y,y';h)\, (1 + \mathcal{O}(|\mathrm{Im}\, t|),\end{aligned}$$

and $b \sim \sum_{j=0}^\infty b_j h^j$ with

$$b_0(t(\tau);y,y') = \langle \nu_y, d(y,t(\tau))\rangle\, \langle \nu_{y'}, d(y',t(\tau))\rangle$$

and $J(t;y,y')$ is the Jacobian in (118). Here, $\beta_1(\tau_1,\tau_2,s) = \alpha_1(\tau_1,\tau_2) + \mathcal{O}(|y(s)-\tau_1|)$ and $\beta_2(\tau_1,\tau_2,s') = \alpha_1(\tau_1,\tau_2) + \mathcal{O}(|y(s')-\tau_1|)$ with $\beta_1 = \beta_2 + \mathcal{O}|y-y'|)$.

Proof. The lemma follows from Lemma 9.2 after using the Morse lemma to make another change of variables $\tau_1 \mapsto \tau_1 + \mathcal{O}(|\tau_1 - y|^2 + |\tau_1 - y'|^2)$. To simplify notation, we continue to denote the new coordinate by τ_1. □

Since

$$\mathrm{Im}\, \partial_{\tau_2}[i\,(y-y')\tau_2 - \beta_1(y-\tau_1)^2 - \beta_2(y'-\tau_1)^2] = i(y-y'),$$

it follows by an integration by parts in τ_2 in Lemma 9.3 that for any $\varepsilon_\circ > 0$

$$\begin{aligned}&N_1^{11}(h)(y,y';a) \\ =&(2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left[i\,(y-y')\tau_2 - \beta_1(y-\tau_1)^2 - \beta_2(y'-\tau_1)^2\right]/h \;\; a(t(\tau_1,\tau_2)) \\ &\times \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y|}{|\mathrm{Im}\, t|}\right) \tilde{\chi}\left(\frac{|\mathrm{Re}\, t - y'|}{|\mathrm{Im}\, t|}\right) \rho(\tau_1,\tau_2;y,y';h')\, \chi(h^{-1+\varepsilon_\circ}|y-y'|) d\tau_1\, d\tau_2 + \mathcal{O}(h^\infty).\end{aligned} \tag{119}$$

Next, we apply steepest descent to the iterated τ_1 Laplace integral

$$I_1(y,y',\tau_2;h) := \int_{\mathbb{R}} e^{-[\beta_1(y-\tau_1)^2 + \beta_2(y'-\tau_1)^2]/h} \rho(\tau_1,\tau_2;y,y',h)\, a(t(\tau_1,\tau_2))\, d\tau_1. \tag{120}$$

Since, $\beta_1 = \beta_2 + \mathcal{O}(y-y')$, the critical points of the phase are

$$\tau_{1,c}(y,y') = \frac{y+y'}{2} + \mathcal{O}(|y-y'|^2).$$

Consequently, by steepest descent, for h sufficiently small,

$$
\begin{aligned}
& I_1(y,y',\tau_2;h) \\
=& (2\pi h)^{1/2} e^{-[\beta_1|y-y'|^2+\mathcal{O}(|y-y'|^3)]/h}\, a(t(y,\tau_2))\,\rho(y,\tau_2;y,y;h) \\
& \times (1+\mathcal{O}(|y-y'|)+\mathcal{O}(h))\cdot\tilde{\chi}^2\left(\frac{|y-y']|}{|\tau_2|}\right)\tilde{\chi}(h^{-1+\varepsilon_\circ}|y-y'|) \\
=& (2\pi h)^{1/2} a(\mathrm{Re}\, t(y,\tau_2),\mathrm{Im}\, t(y,\tau_2)\,\rho(y,\tau_2;y,y;h)\,(1+\mathcal{O}(|y-y'|)+\mathcal{O}(h))\,\chi(h^{-1+\varepsilon_\circ}|y-y'|),
\end{aligned}
\tag{121}
$$

since $\tau_2 \geq \varepsilon_\circ > 0$ on *supp* a. Substitution of (121) in (119) gives

$$
\begin{aligned}
N_1^{11}(h)(y,y';a) = (2\pi h)^{-1/2} \int_{\mathbb{R}} & e^{i(y-y')\tau_2/h}\, e^{-[\beta_1|y-y'|^2+\mathcal{O}(|y-y'|^3)]/h} a(\mathrm{Re}\, t(y,\tau_2),\mathrm{Im}\, t(y,\tau_2)) \\
& \times\rho(y,\tau_2,y,y;h)\,(1+\mathcal{O}(h))\,\chi(h^{-1+\varepsilon_\circ}|y-y'|)\,d\tau_2.
\end{aligned}
\tag{122}
$$

Since $e^{-x^2} = 1+\mathcal{O}(x^2)$ as $x \to 0$, it follows by Taylor expansion of the amplitude term $e^{-[\beta_1|y-y'|^2+\mathcal{O}(|y-y'|^3)]/h}$ in (122) and integration by parts in τ_2 in (122) that

$$
N_1^{11}(h)(y,y';a) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{i(y-y')\tau_2/h}\, a(\mathrm{Re}\, t(y,\tau_2),\mathrm{Im}\, t(y,\tau_2))\,(1+\mathcal{O}(h))\,\rho(y,\tau_2,y,y;h)\,d\tau_2.
\tag{123}
$$

9.3.1. *Identification of $H^{\mathbb{C}}_{\varepsilon_\circ}$ with a subdomain of $B^*\partial\Omega$.* We collect here the explicit formulas identifying $H^{\mathbb{C}}_{\varepsilon_\circ} = q^{\mathbb{C}}(H^{\mathbb{C}}_{\varepsilon_\circ})$ with a subset of $B^*\partial\Omega$. Specifically, given $(\mathrm{Re}\, t, \mathrm{Im}\, t) \in S_{\varepsilon_\circ,\pi}$, it follows from (103) that for the frequency variable $\sigma \in B^*_s\partial\Omega$,

$$
\begin{aligned}
\sigma &= \partial_{s'}\mathrm{Im}\,[i\Psi](\mathrm{Re}\, t,\mathrm{Im}\, t;s,s')|_{s'=s} \\
&= -\langle T_{\partial\Omega}(s), d(s,y(s))\rangle + d_s y(s)\frac{\kappa_H^2(y(s))}{2}|\mathrm{Im}\, t|^2 + \mathcal{O}(|\mathrm{Im}\, t|^3)+\mathcal{O}(|s-s'|^2).
\end{aligned}
\tag{124}
$$

As for the spatial variable $s \in [-\pi,\pi]$, from (114),

$$
y(s) = f(\mathrm{Re}\, t,\mathrm{Im}\, t) = \mathrm{Re}\, t(1+\mathcal{O}(|\mathrm{Im}\, t|)).
\tag{125}
$$

Again, from (124) it is clear that $|\sigma| < 1$ when $\varepsilon_\circ > 0$ is sufficiently small.

Definition 9.4. *We define the* glancing symbol relative to H *associated with* $a(\mathrm{Re}\, t,\mathrm{Im}\, t) \in C_0^\infty(H^{\mathbb{C}}_{\varepsilon_\circ})$ *to be* $a_{\mathcal{G}}(s,\sigma) \in C_0^\infty(\partial\Omega)$ *with*

$$
a_{\mathcal{G}}(s,\sigma) := a(\mathrm{Re}\, t(s,\sigma),\mathrm{Im}\, t(s,\sigma)) \times \rho(y(s),\eta(s,\sigma),y(s),y(s);0),
\tag{126}
$$

where ρ is the function given in Lemma 9.3.

Then, from (123), by L^2-boundedness and the fact that $|b_0(y,\mathrm{Re}\, t(y,\eta))|^2 = \gamma^2(y,\eta) = (1-|\eta|^2)$, we have

$$
h^{-1/2}N_1^{11}(h;a) = Op_h(a_{\mathcal{G}}) + \mathcal{O}(h)_{L^2\to L^2}.
\tag{127}
$$

Moreover, since $\langle \nu_y, d(y,\mathrm{Re}\, t)\rangle = \gamma(y(s), d_s y(s)^{-1}\sigma)$ it follows from Lemma 9.3 that

$$
a_{\mathcal{G}}(s,\sigma) = \frac{1}{\sqrt{2}} a(\mathrm{Re}\, t(s,\sigma),\mathrm{Im}\, t(s,\sigma))\,\kappa_H^{-2}(y(s))\,|\mathrm{Im}\, t(s,\sigma)|^{-1}\,\gamma^2(y(s), d_s y(s)^{-1}\sigma).
\tag{128}
$$

Thus, from (124) and (125), it follows that $a_{\mathcal{G}} \in C_0^\infty(B^*\partial\Omega)$ with

$$a_{\mathcal{G}}(s,\sigma) \geq \frac{1}{C} > 0 \tag{129}$$

when $(\operatorname{Re} t(s,\sigma), \operatorname{Im} t(s,\sigma)) \in \operatorname{supp} a \subset \{q^{\mathbb{C}}(t) \in H^{\mathbb{C}}_{\varepsilon_\circ}; \frac{\varepsilon_\circ}{6} < |\operatorname{Im} t| < \frac{2\varepsilon_\circ}{3}\}$.

9.3.2. *Analysis of the $N_1^{22}(h;a)$-term.* We now estimate the contribution to $N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h)$ coming from $N_1^{22}(h;a)$ where, we recall that

$$\begin{aligned} &N_1^{22}(h)(y,y';a) \\ &= \int\!\!\int_{S_{\varepsilon_\circ,\pi}} N_1^{\mathbb{C}}(h)^*(y,t)\; a(t)\; (1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y|}{|\operatorname{Im} t|}\right)(1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y'|}{|\operatorname{Im} t|}\right) \times N^{\mathbb{C}}(h)(t,y')\, dt\, d\bar{t} \\ &= (2\pi h)^{-1} \int\!\!\int_{S_{\varepsilon_\circ,\pi}} e^{i\Psi(s,s',t)/h}\; a(t)\; (1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y(s)|}{|\operatorname{Im} t|}\right)(1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y'(s)|}{|\operatorname{Im} t|}\right) dt d\bar{t}. \end{aligned} \tag{130}$$

So, in this case we restrict to the range

$$|\operatorname{Im} t| \leq \min(|\operatorname{Re} t - y(s)|, |\operatorname{Re} t - y'(s)|) \tag{131}$$

in the Taylor expansions (101) and (102) of the phase function, $i\Psi(t,s,s')$. In addition, we have the constraint (coming from the definition of $N_1^{\mathbb{C}}(h)$ in (82)) that

$$\max(|t-y(s)|, |t-y'(s)|) \lessapprox \varepsilon_\circ. \tag{132}$$

Then, for $\operatorname{Im} t \geq 0$,

$$\begin{aligned} &\operatorname{Re} i\Psi(t,s,s') = -\left[\, \kappa_H(y)^2(\operatorname{Re} t - y(s))^2 + \kappa_H(y')^2(\operatorname{Re} t - y'(s))^2 \,\right] \operatorname{Im} t \\ &+\mathcal{O}(|\operatorname{Re} t - y(s)|^\alpha |\operatorname{Im} t|^\beta) + \mathcal{O}(|\operatorname{Re} t - y'(s)|^\alpha |\operatorname{Im} t|^\beta) \end{aligned} \tag{133}$$

where, $\alpha+\beta \geq 4$ and $\beta \geq 1$. Substituting the constraints in (131) and (132) in (133) implies that

$$\operatorname{Re}[i\Psi](t,s,s') \leq -2\kappa_H^2 |\operatorname{Im} t|^3 + \mathcal{O}(\varepsilon_\circ)|\operatorname{Im} t|^3 \leq -C(\varepsilon_\circ)|\operatorname{Im} t|^3, \tag{134}$$

with $C(\varepsilon_\circ) > 0$ provided $\varepsilon_\circ > 0$ is sufficiently small. Substitution of the phase bound in (134) in the Schwartz kernel formula in (130) gives

$$\begin{aligned} &|N_1^{22}(h)(y,y',a)| \\ &\leq (2\pi h)^{-1} \int\!\!\int_{S_{\varepsilon_\circ,\pi}} e^{\operatorname{Re}[i\Psi](t,s,s')/h}\, |a(t)|\, (1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y(s)|}{|\operatorname{Im} t|}\right)(1-\tilde{\chi})\left(\frac{|\operatorname{Re} t - y'(s)|}{|\operatorname{Im} t|}\right) dt d\bar{t} \\ &\leq (2\pi h)^{-1} \int\!\!\int_{S_{\varepsilon_\circ,\pi}} e^{-C(\varepsilon_\circ)|\operatorname{Im} t|^3/h}\, |a(t)|\, dt d\bar{t} \\ &= \mathcal{O}(h^{-1} e^{-C(\varepsilon_\circ)\,\varepsilon_\circ^3/h}), \end{aligned}$$

since by assumption $\operatorname{supp} a \subset H^{\mathbb{C}}_{\varepsilon_\circ} \cap \{t; \frac{\varepsilon_\circ}{6} < \operatorname{Im} t < \frac{2\varepsilon_\circ}{3}\}$. Consequently, the $N_1^{22}(h)$-term is exponentially decaying in h and is negligible.

10. Mixed terms

In the following we continue to write $L^2 := L^2(\partial\Omega)$. Then, by our proof of Proposition 8.1 in section 9, there is a decomposition $N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h) = N_1^{11}(h;a) + N_1^{22}(h;a) + N_1^{21}(h;a) + N_1^{12}(h;a)$, where

$$N_1^{11}(h;a) = h^{1/2} Op_h(a_{\mathcal{G}}) + \mathcal{O}(h)_{L^2\to L^2},$$
$$\|N_1^{22}(h;a)\|_{L^2\to L^2} = \mathcal{O}(h^{-1}e^{-C(\varepsilon_\circ)/h}),$$

with $C(\varepsilon_\circ) > 0$ and

$$\|N_1^{12}(h;a)^* N_1^{12}(h;a)\|_{L^2\to L^2} = \|N_1^{11}(h;a)^* N_1^{22}(h;a)\|_{L^2\to L^2} = \mathcal{O}(h^{-1/2}e^{-C(\varepsilon_\circ)/h}).$$

The same estimate holds for $N_1^{21}(h;a)^* N_1^{21}(h;a)$. As a result,

$$N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h) = h^{1/2} Op_h(a_{\mathcal{G}}) + \mathcal{O}(h). \tag{135}$$

So, in particular,

$$\|e^{-S/h} N_1^{\mathbb{C}}(h)\|_{L^2(\partial\Omega)\to L^2(\operatorname{supp} a)} = \mathcal{O}(h^{1/2}). \tag{136}$$

From the "far-diagonal" bound in (98)

$$\|N_2^{\mathbb{C}}(h)^* e^{-2S/h} a N_2^{\mathbb{C}}(h)\|_{L^2\to L^2} = \mathcal{O}(e^{-C(\varepsilon_\circ)/h}).$$

Thus,

$$\|e^{-S/h} N_2^{\mathbb{C}}(h)\|_{L^2(\partial\Omega)\to L^2(\operatorname{supp} a)} = \mathcal{O}(e^{-C(\varepsilon_\circ)/2h}). \tag{137}$$

From (136) and (137) it then follows by Cauchy-Schwarz that the mixed terms

$$\max(\|N_2^{\mathbb{C}}(h)^* e^{-2S/h} a N_1^{\mathbb{C}}(h)\|_{L^2\to L^2}, \|N_1^{\mathbb{C}}(h)^* e^{-2S/h} a N_2^{\mathbb{C}}(h)\|_{L^2\to L^2}) = \mathcal{O}(e^{-C'(\varepsilon_\circ)/h}) \tag{138}$$

with $C'(\varepsilon_\circ) > 0$. So, (135) and (138) imply that for $h \in (0, h_0(\varepsilon_\circ)]$ with $h_0 > 0$ sufficiently small,

$$N^{\mathbb{C}}(h)^* e^{-2S/h} a N^{\mathbb{C}}(h) = h^{1/2} Op_h(a_{\mathcal{G}}) + \mathcal{O}(h)_{L^2\to L^2}.$$

This finishes the proof of Proposition 8.1 and Proposition 1.2 is an immediate consequence in view of (129). ☐

11. Analysis near corner points

We assume now that $\Omega \subset \mathbb{R}^2$ is a smooth domain with corners. In general, we define a smooth domain with corners in $\mathbb{R}^n$ and with M boundary faces (hypersurfaces) to be a set of the form $\{x \in \mathbb{R}^n : \rho_j(x) \leq 0, j = 1, \dots, M\}$, where the defining functions ρ_j are smooth in a neighborhood of Ω, and are independent in the following sense: at each p such that $\rho_j(p) = 0$ for some finite subset $j \in I_p$ of indices, the differentials $d\rho_j$ are linearly independent for all $j \in I_p$. A boundary hypersurface H_j is the intersection of Ω with one of the hypersurfaces $\{\rho_j = 0\}$. The boundary faces of codimension k are the components of $\rho_{j_1} = \cdots = \rho_{j_k} = 0$ for some subset $\{j_1, \dots, j_k\}$ of the indices; each is a manifold with corners. In Theorem 1.2 it is essential to allow Ω to have corners since domains with ergodic billiard flow in $\mathbb{R}^n$ have corners. The singularities play a very significant role in the dynamics.

We denote the smooth part of $\partial\Omega$ by $(\partial\Omega)^o$. Here, and throughout this article, we denote by W^o the interior of a set W and, when no confusion is possible, we also use it to denote the regular set of $\partial\Omega$. Thus, $\partial\Omega = (\partial\Omega)^o \cup \Sigma$, where $\Sigma = \bigcup_{i\neq j}(W_i \cap W_j)$ is the singular set. When $\dim\Omega = 2$, the singular set is a finite set of points and the W_i are smooth curves. In higher dimensions, the W_i are smooth hypersurfaces; $W_i \cap W_j$ is a stratified smooth space of

co-dimension one, and in particular Σ is of measure zero. We denote by $S^*_\Sigma\Omega$ the set of unit vectors to Ω based at points of Σ. We also define $C^\infty(\partial\Omega)$ to be the restriction of $C^\infty(\mathbb{R}^n)$ to $\partial\Omega$. We define the open unit ball bundle $B^*(\partial\Omega)^o$ to be the projection to $T^*\partial\Omega$ of the inward pointing unit vectors to Ω along $(\partial\Omega)^o$. We leave it undefined at the singular points.

For concreteness, here we assume $n = 2$ and write the smooth part of the boundary as a disjoint union $(\partial\Omega)^o = \bigcup_{j=1}^M W_j^o$, where the W_j^o are open boundary faces diffeomorphic to open intervals of $\mathbb{R}$. We let $r_j : (a_j, a_j + 1) \to W_j^o,\ s \mapsto r_j(s)$ denote unit-speed parametrizations of the boundary faces with $a_0 = -\pi, a_M = \pi$ and let

$$y : (a_j, a_{j+1}) \to H_j \subset H,\ s \mapsto y(s),$$

be the parametrization defining the glancing set relative to H_j. Here, the H_j's are just open sub-arcs of H. For fixed small $\varepsilon_\circ > 0$ let $\chi_j^{\varepsilon_\circ} \in C_0^\infty(\mathbb{R})$ be a cutoff equal to 1 on $(a_j + \varepsilon_\circ, a_{j+1} - \varepsilon_\circ)$ for some boundary face indexed by $j \in \{1, ..., M\}$ It follows that

$$\int\int_{S_{\varepsilon_\circ,\pi}} e^{-2S(t)/h} |N^{\mathbb{C}}(h)\chi_j^{\varepsilon_\circ}\varphi_h^{\partial\Omega}(t)|^2\, \chi_{\varepsilon_\circ}(t)\, dtd\bar{t} \tag{139}$$

$$= (2\pi h)^{-1} \int\int_{S_{\varepsilon_\circ,\pi}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\Psi(t,s,s')/h} \chi_j^{\varepsilon_\circ}(s)\chi_j^{\varepsilon_\circ}(s')\, \varphi_h^{\partial\Omega}(s)\overline{\varphi_h^{\partial\Omega}(s')}\, dsds' \right) dtd\bar{t} \tag{140}$$

The analysis of the last integral on the RHS of (139) follows exactly as in section 9 and one gets that

$$\int\int_{S_{\varepsilon_\circ,\pi}} e^{-2S(t)/h} N^{\mathbb{C}}(h)\chi_j^{\varepsilon_\circ}\varphi_h^{\partial\Omega}(t) \cdot \overline{N^{\mathbb{C}}(h)\chi_k^{\varepsilon_\circ}\varphi_h^{\partial\Omega}(t)}\, \chi_{\varepsilon_\circ}(t)\, dtd\bar{t} = \langle Op_h(a_{\mathcal{G}}^{(j)})\varphi_h^{\partial\Omega}, \varphi_h^{\partial\Omega}\rangle_{L^2},$$

where, $\text{supp} a_{\mathcal{G}}^{(j)} \in C_0^\infty(B^*W_j^o)$ with $\int_{B^*W_j^o} a_{\mathcal{G}}^{(j)}(s,\sigma)\gamma(s,\sigma)dsd\sigma > 0$. Thus, Proposition 8.1 follows also in the case where $\partial\Omega$ is only piecewise smooth. Theorem 1.2 then follows form Theorem 1.1 as outlined in the introduction. ☐

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